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Senior Math Circles
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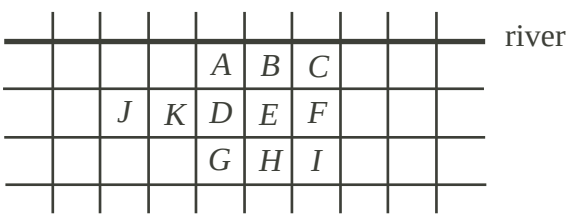
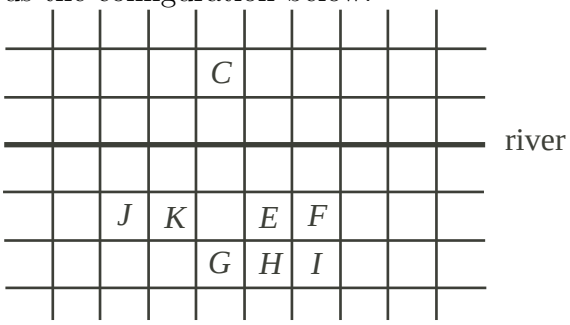
Game Theory III

G1. *(Conway's Checker Jumping)* Imagine an infinite checkerboard with a fixed horizontal line, which we will call the *river*. A checker is placed on every square below the river. You can jump checkers over one another horizontally or vertically (but not diagonally) if the destination square is empty. However, each time you do so, the checker that is jumped over is removed from the board.
We can immediately get a checker on the first row beyond the river by making one jump up the board. From here, we see that it is possible to get a checker on the second row beyond the river by making two more jumps.

Is it possible to get a checker on the third row beyond the river?



In the above diagram we can jump D up to get a checker on the first row past the river. Then we jump C left, then up (over D) to get a checker on the second row beyond the river, giving us the configuration below.



From here, we jump J right, G up, F left, F up and F up again to get a checker on the third row.

Is it possible to get a checker on the fourth row beyond the river?

Yes it is possible, however we will not show it here as it takes a lot of moves.

Can we get a checker as far as we want past the river?

No! Surprisingly, as proved by John Horton Conway, we can only get as far as the fourth row past the river.

Proof:

The Idea:
Define an "energy" function on each square. If a piece is on that square, we have that amount of energy. Furthermore, make sure the total energy never increases when we make a jump. If the destination square has energy 1 and any *finite* sum of starting squares has energy less than 1, then we have proven we cannot get to the destination, since we do not have enough energy.

Define the distance from destination square as the number of horizontal/vertical steps away from the destination square.
For example, if X is the destination square on the fifth row above the river, then the distance of the squares around it are shown in the diagram below.

4	3	2	1	X	1	2	3	4
5	4	3	2	1	2	3	4	5
6	5	4	3	2	3	4	5	6
7	6	5	4	3	4	5	6	7
8	7	6	5	4	5	6	7	8
9	8	7	6	5	6	7	8	9
10	9	8	7	6	7	8	9	10

river

The further away we are, the less energy we should have, so let us define:
Energy of Square $= \varphi^{(DistanceAway)}, \varphi \in (0, 1)$
As distance away increases in this formula, the energy decreases. Furthermore, this satisfies the destination square having energy 1.

Now, how do we choose φ? To make sure the total energy never increases, we need to make sure $\varphi^{n+2} + \varphi^{n+1} \geq \varphi^n$ for $n \geq 0$. Equivalently, we need $\varphi^2 + \varphi \geq 1$. Solving we get $\varphi \geq 0.6180339 = \phi - 1$, where $\phi = \dfrac{1+\sqrt{5}}{2}$ is the Golden Ratio.

How much energy do we initially have?
Note we have a checker on:
1 square at distance 5,
3 squares at distance 6,
5 squares at distance 7,
7 squares at distance 8,
and so on.

Hence our total energy is $\varphi^5 + 3\varphi^6 + 5\varphi^7 + 7\varphi^8 + \cdots$.
To maximize the tightness of the argument, we need to minimize the starting energy, so choose

$\varphi = \phi - 1$, and remember that this means $\varphi^2 + \varphi = 1$.

To solve this, we must solve an Arithmetic-Geometric Series. To do so, let

$$\begin{array}{rrl} & S & = \varphi^5 + \quad 3\varphi^6 + 5\varphi^7 + 7\varphi^8 + \cdots \\ - & \varphi S & = \qquad\quad \varphi^6 + 3\varphi^7 + 5\varphi^8 + \cdots \\ \hline & (1-\varphi)S & = \varphi^5 + \quad 2\varphi^6 + 2\varphi^7 + 2\varphi^8 + \cdots \end{array}$$

$$\begin{aligned} \varphi^2 S &= \varphi^5 + 2\varphi^6(1 + \varphi + \varphi^2 + \cdots) \\ \varphi^2 S &= \varphi^5 + 2\varphi^6 \left(\tfrac{1}{1-\varphi}\right) \\ \varphi^2 S &= \varphi^5 + 2\varphi^4 \\ S &= \varphi^3 + 2\varphi^2 \\ &= \varphi^2 + \varphi \\ &= 1 \end{aligned}$$

The <u>infinite</u> checkerboard only has energy 1, so any finite subset of it has energy less than 1. Therefore, we can never get to the 5th row beyond the river because we do not have enough energy.

G2. *(Intelligent Pirates Problem)* There are 5 greedy pirates who stumbled across 100 gold pieces, and each one is trying to maximize the amount they keep to themself. Their negotiation algorithm works as follows:

- the pirates line up in order
- the first pirate proposes a distribution of the gold pieces amongst the pirates (it is possible to propose that some pirate gets zero gold pieces)
- all of the pirates, *including the pirate who made the proposal*, cast a vote either for or against the distribution
- if half *or more* of the pirates agree with the distribution, then the distribution happens and we are finished
- otherwise, the pirate who made the proposal is beheaded and the next pirate in line makes a proposal, and so on

Assume that all the pirates have infinite intelligence, that is, they can foresee the optimal strategies of each other and cast their votes accordingly to maximize their individual yield. Furthermore, the pirates are evil, so when faced with two choices which result in getting the same amount of gold pieces, they will always make the choice that kills more of the other pirates.
Working backwards, deduce how the first pirate should split the gold pieces to maximize his individual yield.

Solution: We will work backwards.
Consider what happens if Pirates 1, 2 and 3 are killed and Pirate 4 proposes a division of gold. He will propose 100 gold pieces for himself, and 0 for Pirate 5. Then he will vote yes.

Regardless of what Pirate 5 does, the proposal will go through and Pirate 5 will get no gold.

Therefore Pirate 5 will do everything he can to make sure Pirate 4 never gets to make a proposal. Thus if Pirate 3 offers him at least 1 gold, he will agree with the proposal. Hence if Pirates 1 and 2 are killed and Pirate 3 is allowed to make a proposal, he can propose 99 to himself, 0 to Pirate 4 and 1 to Pirate 5, and both Pirate 3 and Pirate 5 will agree.

Thus if Pirate 1 is killed and so it is Pirate 2's turn to make a proposal, he can propose 99 to himself and 1 to Pirate 4 and Pirate 4 will agree since Pirate 4 would get 0 otherwise (if Pirate 3 was allowed to make a proposal).

Finally, we arrive at the current situation, with Pirate 1 making a proposal. He knows Pirate 3 will vote yes if offered only 1 gold piece. He needs one more pirate to vote yes. Pirate 2 will be impossible to please since he would have a chance to make a proposal next. Pirate 5 will receive 0 gold pieces if Pirate 2 is allowed to make a proposal, so he will vote yes if offered one gold piece.

Therefore, Pirate 1's optimal strategy is to give 1 gold piece to Pirate 3 and 1 gold piece to Pirate 5, keeping 98 gold pieces for himself.

G3. *(Nash Equilibrium)* Alice and Bob are playing rock-paper-scissors. However, Alice is handicapped and unable to clench her first, hence she can never play rock.
Bob can never lose by always playing scissors. However, Alice, being a smart girl, will then always play scissors as well, so that Bob can never win. Clearly Bob has the advantage, so he is not willing to settle for this. He will sometimes play rock to beat Alice's scissors. However, then Alice will counter by occasionally playing paper. Determine how frequently Bob should play rock, and how frequently Alice should play paper.

Solution:

Clearly Bob should never play paper, since it will never win. Suppose he plays rock a fraction r of the time and scissors a fraction s of the time with $r + s = 1$. He knows Alice will play optimally given his choice of r, s. What will she do?

Alice: $E(\text{scissors}) = r(-1) + s(0) = -r$
$E(\text{paper}) = r(1) + s(-1) = r - s = 2r - 1$
Thus Alice will play scissors if and only if $-r \geq 2r - 1$, i.e. $r \leq \frac{1}{3}$.

Hence if $r \leq \frac{1}{3}$, Bob's Expectation is r. The maximum value is $\frac{1}{3}$ when $r = \frac{1}{3}$. If $r \geq \frac{1}{3}$, Bob's Expectation is $-r + s = 1 - 2r$. The maximum value of this is $\frac{1}{3}$ when $r = \frac{1}{3}$.
Thus Bob's Expectation is $\frac{1}{3}$ each game by choosing $r = \frac{1}{3}$. He should win 2 out of 3 times (not counting ties). Note that Bob made the choice that made it impossible for Alice to use her intelligence. Regardless of whether she chooses paper or scissors, her chances of winning are the same.

Now let's look at the game from Alice's perspective. She knows Bob is also intelligent. She must choose how often to play scissors and paper so that Bob doesn't "exploit" her. Suppose she plays scissors a fraction s of the time and paper a fraction p of the time, with $s + p = 1$. She now considers what Bob does:
Bob: $E(\text{rock}) = s(1) + p(-1) = s - p = 2s - 1$
$E(\text{scissors}) = s(0) + p(1) = p = 1 - s$
Thus Bob will play rock if and only if $2s - 1 \geq 1 - s$ i.e. $s \geq \frac{2}{3}$.

Hence, if $s \geq \frac{2}{3}$, Alice's Expectation is $1 - 2s$. The maximum value of this is $-\frac{1}{3}$ when $s = \frac{2}{3}$. If $s \leq \frac{2}{3}$, Alice's Expectation is $s - 1$. The maximum value of this is $-\frac{1}{3}$ when $s = \frac{2}{3}$. Therefore, Alice's optimal strategy is playing scissors $\frac{2}{3}$ of the time and paper $\frac{1}{3}$ of the time.

The strategies are represented in the table below.

		Alice	
		Scissors $\left(\frac{2}{3}\right)$	Paper $\left(\frac{1}{3}\right)$
Bob	Scissors $\left(\frac{2}{3}\right)$	Tie	Bob Wins
	Rock $\left(\frac{1}{3}\right)$	Bob Wins	Alice Wins

Therefore, $\frac{4}{9}$ of the time Alice and Bob tie, $\frac{4}{9}$ of the time Bob wins and $\frac{1}{9}$ of the time Alice wins.

The above was an example of an optimization problem. This particular type of optimization problem has applications in elections, economics, games involving betting ("bluffing" vs. "value-betting"), the game "chicken" and The Prisoner's Dilemna.

Problem Set

1. Consider G1 above.

   (a) Prove that it is impossible to get a checker on the second row beyond the river in less than 3 moves.

   (b) Prove that it is impossible to get two adjacent checkers on the fourth row beyond the river.

   (c) If we allow diagonal jumps, find a way to get a checker on the eighth row beyond the river.

   (d) Prove that it is impossible to get a checker on the ninth row beyond the river, even if we allow diagonal jumps.

   (e) Suppose we had the same setup on a 3-dimensional checkerboard (now the river is a fixed plane). Prove that it is impossible to get a checker on the eighth plane beyond this fixed plane. (Note: It is in fact possible to get a checker on the seventh plane beyond the fixed plane, but the visualization is extremely difficult.)

2. Consider G2 above.

   (a) What if we had 6 pirates? 7 pirates? Deduce a general pattern, assuming we had enough gold pieces.

   (b) If there were 23 pirates and only 10 gold pieces, prove that the first pirate has no method of survival.

3. Consider G3 above.
   Alice and Bob are playing a game where they each choose A or B secretly with the following payoff matrix. (The numbers in the grid are the amount Alice gives to Bob, so a negative number indicates Bob is giving money to Alice.) Find their optimal strategies.

Some Solutions

1. (d) If we allow diagonal jumps, then we must modify our distance away function to allow for diagonal steps. The diagram below shows this new distance function if X is the destination square.

	3	2	1	X	1	2	3
	3	2	1	1	1	2	3
	3	2	2	2	2	2	3
	3	3	3	3	3	3	3

So if X is in row n above the river, our total starting energy is

$$\begin{array}{rrl} & S & = (2n+1)\varphi^n + \quad (2n+3)\varphi^{n+1} + (2n+5)\varphi^{n+2} + \cdots \\ - & \varphi S & = \qquad\qquad\qquad (2n+1)\varphi^{n+1} + (2n+3)\varphi^{n+2} + \cdots \\ \hline & (1-\varphi)S & = (2n+1)\varphi^n + \quad 2\varphi^{n+1}(1+\varphi+\varphi^2+\cdots) \\ & \varphi^2 S & = (2n+1)\varphi^n + \quad 2\varphi^{n-1} \end{array}$$

Note that if $n = 9$, we have $S = 19\varphi^7 + 2\varphi^6 < 1$.
If $n = 8$, we have $S = 17\varphi^6 + 2\varphi^5 > 1$.

Therefore we have proven that we cannot get to the 9th row. We cannot prove that we cannot get to the 8th row, however simply because the energy is greater than 1 for $n = 8$ does not prove we *can* get to the 8th row.

There is in fact a sequence of moves, which you could find with some trial and error, that would get a checker to the 8th row.

(e) If we add a third dimension the total initial energy is
$S = \varphi^n + 5\varphi^{n+1} + 13\varphi^{n+2} + 25\varphi^{n+3} + 41\varphi^{n+4}$

$$S = \varphi^n \sum_{k=0}^{\infty} \varphi^k (1 + 2k(k+1))$$

$$S = \varphi^n \left(\sum_{k=0}^{\infty} \varphi^k + 2\sum_{k=0}^{\infty} k\varphi^k + 2\sum_{k=0}^{\infty} k^2\varphi^k \right)$$

We know that $\displaystyle\sum_{k=0}^{\infty} \varphi^k = \frac{1}{1-\varphi} = \frac{1}{\varphi^2}$

Let

$$\begin{array}{rrl} & Q = \displaystyle\sum_{k=0}^{\infty} k\varphi^k & = \varphi \quad +2\varphi^2 + 3\varphi^3 + \cdots \\ - & \varphi Q & = \qquad \varphi^2 + 2\varphi^3 + \cdots \\ \hline & (1-\varphi)Q & = \varphi \quad +\varphi^2 + \varphi^3 + \cdots \end{array}$$

$$\begin{aligned}
\varphi^2 Q &= \varphi\left(\frac{1}{1-\varphi}\right) \\
Q &= \frac{1}{\varphi^3}
\end{aligned}$$

Let

$$\begin{array}{rrcl}
 & T = \sum_{k=0}^{\infty} k^2\varphi^k & = \varphi & +4\varphi^2 + 9\varphi^3 + \cdots \\
- & \varphi T & = & \varphi^2 + 4\varphi^3 + \cdots \\
\hline
 & (1-\varphi)T & = \varphi & +3\varphi^2 + 5\varphi^3 + \cdots
\end{array}$$

$$\begin{aligned}
\varphi^2 T &= \sum_{k=0}^{\infty}(2k+1)\varphi^{k+1} \\
\varphi^2 T &= \varphi\sum_{k=0}^{\infty}\varphi^k + 2\varphi\sum_{k=0}^{\infty}k\varphi^k \\
\varphi^2 T &= \frac{\varphi}{\varphi^2} + \frac{2\varphi}{\varphi^3} \\
\varphi^2 T &= \frac{\varphi+2}{\varphi^2} \\
T &= \frac{\varphi+2}{\varphi^4}
\end{aligned}$$

Finally we have

$$\begin{aligned}
S &= \varphi^n\left(\frac{1}{\varphi^2} + \frac{2}{\varphi^3} + \frac{2(\varphi+2)}{\varphi^4}\right) \\
&= \varphi^n\left(\frac{\varphi^2 + 2\varphi + 2\varphi + 4}{\varphi^4}\right) \\
&= \varphi^n\left(\frac{3\varphi+5}{\varphi^4}\right) \\
&= \varphi^n\left(\frac{2+\frac{3}{\varphi}}{\varphi^4}\right) \\
&= \varphi^n\left(\frac{\frac{1}{\varphi}+\frac{2}{\varphi^2}}{\varphi^4}\right) \\
&= \varphi^n\left(\frac{\frac{1}{\varphi^2}+\frac{1}{\varphi^3}}{\varphi^4}\right) \\
&= \frac{\varphi^n}{\varphi^8}
\end{aligned}$$

Therefore, if we choose $n = 8$, the total energy is equal to 1, so we cannot get to the 8th level above the fixed plane.