Intermediate Math Circles  
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Solving Linear Diophantine Equations

Diophantine equations are equations intended to be solved in the integers. For example, Fermat’s Last Theorem is the statement that if \( n \geq 3 \), the equation

\[
x^n + y^n = z^n
\]

has no solution in the integers, except the solutions with one of \( x, y, \) or \( z \) being 0.

It’s too hard to try to understand all diophantine equations in one go, so we’ll look at linear ones, like

\[
7x + 3y = 4
\]

or

\[
10x + 4y = 12.
\]

How can we find solutions to these with \( x \) and \( y \) being integers?

**Example:** Suppose that Bob has $1.55 in quarters and dimes. How many quarters and how many dimes does he have? (There might be more than one solution!)

**Solution:** We’re trying to solve \( 25x + 10y = 155 \) with \( x \) and \( y \) integers (which shouldn’t, in this case, be negative!). By trial-and-error we can find

\[
\begin{align*}
x &= 1 & y &= 13 \\
\text{or} & & x &= 3 & y &= 8 \\
\text{or} & & x &= 5 & y &= 3
\end{align*}
\]

**Example:** A robot can move backwards or forwards with big steps (130 cm) or small steps (50 cm). Is there a series of moves it can make to end up 10 cm ahead of where it started? i.e., can we solve \( 130x + 50y = 10? \)
Trial-and-error solution:

\[ x = 2 \quad y = -5 \]

\[ \rightarrow \rightarrow \rightarrow \quad \text{2 big steps forward} \]

\[ \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \quad \text{5 small steps back} \]

\[ \text{OR} \quad x = 7 \quad y = -18 \]

There are many solutions.

**Main question:** If \( a \), \( b \), and \( c \) are integers, the how can you find a solution to

\[ ax + by = c \]

in integers?

Trick question! There might not be one!

For example, look at the equation

\[ 3x + 6y = 5 \]

If we could find integers \( x \) and \( y \) for which that holds, then

\[ x + 2y = \frac{5}{3} \]

But \( \frac{5}{3} \) is not an integer.

To figure out exactly when this sort of equation can be solved, we need some number theory.

If any integer divides both \( a \) and \( b \), then

\[ ax + by = c \]

can only have a solution if that integer also divides \( c \).

**Remember:** an integer \( d \) “divides” another integer \( e \) if and only if \( \frac{e}{d} \) is an integer.

The “greatest common divisor” of \( a \) and \( b \) is the largest integer that divides \( a \) and divides \( b \). We write it as \( \gcd(a, b) \).

For example:

\[ \gcd(3, 6) = 3 \]

\[ \gcd(10, 15) = 5 \]

\[ \gcd(3, 7) = 1 \]

1 divides everything, so \( \gcd(a, b) \) is always at least 1.
If $ax + by = c$ is going to have a solution, then $\gcd(a, b)$ needs to divide $c$.

What’s the best way to calculate $\gcd(a, b)$?

For small numbers $a$ and $b$ it’s not too hard, but try calculating

$$\gcd(104723, 103093).$$

You can try to find all divisors, and see which ones divide both, but that could take all day!

Here’s and interesting idea: If a number $d$ divides $a$ and $b$, and $q$ is any integer, then $d$ divides $a - qb$. Also, if $e$ divides $a - qb$ and $b$, then $e$ divides $a = (a - qb) + qb$. So for any integer $q$,

$$\gcd(a, b) = \gcd(b, a - qb).$$

Why is this useful?

Suppose we want to calculate

$$\gcd(73, 7)$$

This is the same as $\gcd(7, 73 - q \cdot 7)$ for any integer $q$. Like, for example, $q = 10$. So

$$\gcd(73, 7) = \gcd(7, 3) = 1$$

**Example:** $\gcd(117, 55) = ?$

**Solution:** $117 - 2 \cdot 55 = 7$, so

$$\gcd(117, 55) = \gcd(55, 7).$$

Now, $55 - 7 \cdot 7 = 6$, so

$$\gcd(55, 7) = \gcd(7, 6).$$

Again, $7 - 1 \cdot 6 = 1$, so

$$\gcd(7, 6) = \gcd(6, 1) = 1.$$

So $\gcd(117, 55) = 1$. □

This is the “Euclidean algorithm” for calculating $\gcd(a, b)$.

1. **Step 1:** Arrange things so that $a \geq b$.

2. **Step 2:** Write $a = qb + r$, with $0 \leq r < b$.

3. **Step 3:** If $r = 0$, then $b$ divides $a$, so $\gcd(a, b) = b$. STOP!

   If not then $\gcd(a, b) = \gcd(b, r)$.

4. **Step 4:** Repeat to calculate $\gcd(b, r)$.

Since the numbers get smaller at every stage, you eventually get an answer.
Example: Calculate $\gcd(129, 48)$.

Solution:

\[
\begin{align*}
129 &= 2 \cdot 48 + 33, \text{ so } \quad \gcd(129, 48) = \gcd(48, 33) \\
48 &= 1 \cdot 33 + 15, \text{ so } \quad \gcd(48, 33) = \gcd(33, 15) \\
33 &= 2 \cdot 15 + 3, \text{ so } \quad \gcd(33, 15) = \gcd(15, 3) \\
15 &= 5 \cdot 3 + 0, \text{ so } \quad \gcd(15, 3) = 3 \\
\end{align*}
\]

We have $\gcd(129, 48) = 3$.

\[\square\]

Example: Is there a solution to $129x + 48y = 4$?

Solution: NO! Because $\gcd(129, 48) = 3$, which doesn’t divide 4.

So, we know how to show (sometimes) that $ax + by = c$ has no solution, but if it does have a solution, is there a clever way to find one?

For example, can we find a solution to $117x + 55y = 1$?

(Remember that $\gcd(117, 55) = 1$).

We found that

\[
\begin{align*}
117 - 2 \cdot 55 &= 7 \\
55 - 7 \cdot 7 &= 6 \\
7 - 1 \cdot 6 &= 1 \\
6 - 1 \cdot 6 &= 0
\end{align*}
\]

Let’s rearrange this:

\[
\begin{align*}
1 &= 7 - 1 \cdot 6 \\
6 &= 55 - 7 \cdot 7, \text{ so } \\
1 &= 7 - 1 \cdot (55 - 7 \cdot 7) \\
&= 8 \cdot 7 - 1 \cdot 55 \\
\text{But } 7 &= 117 - 2 \cdot 55, \text{ so } \\
1 &= 8 \cdot (117 - 2 \cdot 55) - 1 \cdot 55 \\
&= 8 \cdot 117 - 17 \cdot 55
\end{align*}
\]

So one solution to $117x + 55y = 1$ is $x = 8$, $y = -17$.

This always works, and gives us a way to find a solution to $ax + by = c$ if $c = \gcd(a, b)$.
**Example**: Find a solution to $4389x + 2919y = 21$.

**Solution**: We can only do this if we happen to have $\gcd(4389, 2919) = 21$.

\[
\begin{align*}
4389 &= 1 \cdot 2919 + 1470 \\
2919 &= 1 \cdot 1470 + 1449 \\
1470 &= 1 \cdot 1449 + 21 \\
1449 &= 69 \cdot 21 + 0
\end{align*}
\]

So $\gcd(4389, 2919) = 21$

\[
\begin{align*}
21 &= 1470 - 1 \cdot 1449 \\
&= 1470 - 1 \cdot (2919 - 1 \cdot 1470) \\
&= 2 \cdot 1470 - 1 \cdot 2919 \\
&= 2 \cdot (4389 - 1 \cdot 2919) - 1 \cdot 2919 \\
&= 2 \cdot 4389 - 3 \cdot 2919
\end{align*}
\]

So one solution is $x = 2$, $y = -3$. □

What about solving $ax + by = c$ when $c$ is not $\gcd(a, b)$?

We know that we need $\gcd(a, b)$ to divide $c$, so $\frac{c}{\gcd(a, b)}$ is an integer.

If $ax + by = \gcd(a, b)$, then

\[
\begin{align*}
ax + by &= \gcd(a, b) \\
&= a \left(x \cdot \frac{c}{\gcd(a, b)}\right) + b \left(y \cdot \frac{c}{\gcd(a, b)}\right) \\
&= \frac{c}{\gcd(a, b)} \cdot (ax + by) \\
&= c
\end{align*}
\]

So just

1. Solve $ax + by = \gcd(a, b)$.

2. Multiply the $x$ and $y$ in the solution by $\frac{c}{\gcd(a, b)}$.

**Example**: Find a solution to $4389x + 2919y = 231$.

**Solution**: We know that $\gcd(4389, 2919) = 21$, so this is only going to work if 21 divides 231. It does!

\[
\frac{231}{21} = 11, \text{ so}
\]

1. Find a solution to $4389x + 2919y = 21$. We’ve done this: $x = 2$, $y = -3$.

2. Multiply by 11. So the new solution is $x = 22$, $y = -33$.

Check: $4389(22) + 2919(-33) = 231$. □
So we now know that

\[ ax + by = c \]

has a solution (in the integers) if and only if \( \gcd(a, b) \) divides \( c \), and we know how to find a solution if there is one.

How can we find more solutions?

Suppose \( x, y \) is a solution to \( ax + by = c \).

For example, \( x = 2, y = -5 \) is a solution to

\[ 18x + 7y = 1 \]

Notice that \( x = 2 + 7 \) and \( y = -5 - 18 \) is also a solution. In fact, if \( k \) is any integer,

\[ x = 2 + 7k, \quad y = -5 - 18k \]

is a solution, since

\[
18(2 + 7k) + 7(-5 - 18k) \\
= 36 + 18 \cdot 7 \cdot k - 35 - 18 \cdot 7 \cdot k \\
= 1
\]

You can use this formula to describe all solutions to \( 18x + 7y = 1 \).

In general, this works for any equation \( ax + by = c \). If \( x = x_0, y = y_0 \) is one solution, then the full set of solutions is given by choosing integers \( k \), and letting \( x = x_0 + k \cdot e, \quad y = y_0 - k \cdot f \), where

\[
e = \frac{a}{\gcd(a, b)} \quad f = \frac{b}{\gcd(a, b)}
\]

**Example:** Write down a formula giving the solutions to

\[ 4389x + 2919y = 231 \]

**Solution:** We already have one: \( x = 22, y = -33 \)

Now, \( e = \frac{a}{\gcd(a, b)} = \frac{4389}{21} = 209 \)

\( f = \frac{b}{\gcd(a, b)} = \frac{2919}{21} = 139 \)

So the solutions are exactly

\[ x = 22 + k \cdot 209 \]
\[ y = -33 - k \cdot 139 \]

for all integers \( k \).
Problems

1. Here’s a little puzzle: start with the number 0, and at every step, you may add or subtract either the number 5 or the number 17 (that’s four possible moves in total). Is it possible to eventually get to the number 1?

\[ 0 \xrightarrow{+17} 17 \xrightarrow{-5} 12 \xrightarrow{} \ldots \]

2. Find the greatest common divisors:
   a) \( \text{gcd}(55, 20) \)
   b) \( \text{gcd}(318, 225) \)
   c) \( \text{gcd}(2009, 4182) \)
   d) \( \text{gcd}(43477, 35021) \)
   e) \( \text{gcd}(127146, 123456) \)
   f) \( \text{gcd}(422058756, 464963310) \)

3. Find a solution to each of the following diophantine equations, or explain where there is none:
   a) \( 55x + 20y = 5 \)
   b) \( 318x + 225y = 4 \)
   c) \( 2009x + 4182y = 820 \)
   d) \( 43477x + 35021y = 7 \)
   e) \( 127146x + 123456y = 12 \)
   f) \( 422058756x + 464963310y = 42 \)

4. For each of the above equations (except those that didn’t have solutions!) write down a formula describing ALL solutions to the equations in terms of an integer \( k \).