From Pascal’s Triangle to Sierpinski’s Triangle

Nicoleta Babutiu
Today’s journey

Intro

Pascal’s Triangle

Sierpiński’s Triangle

Conclusions

Q&A
Do you remember “The 12 Days of Christmas” song?

A partridge in a pear tree
Two turtle doves
Three French hens
Four calling birds
Five gold rings
Six geese a-laying
Seven swans a-swimming
Eight maids a-milking
Nine ladies dancing
Ten lords a-leaping
Eleven pipers piping
Twelve drummers drumming
How many gifts were given in total over the 12 days?

• On the first day: 1

• On the second day: 1 + 2

• On the 3rd day: 1 + 2 + 3

• On the 12th day: 1 + 2 + 3 + … + 12

\[ 1 + (1+2) + (1+2+3) + \ldots + (1+2+3+\ldots+12) = \]
Pascal’s Triangle

Blaise Pascal - “Treatise on Arithmetical Triangle”, 1655

Yang Hui’s Triangle - the 13th century

Tartaglia’s Triangle - in 1556

Number of gifts received each day

Running total number of gifts received each day

Answer: 364
Pascal’s Method

In how many different paths can you spell SIERPINISKI if you start at the top and proceed to the next row by moving diagonally left or right?
Binomial Coefficients

Binomial Theorem

\( (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \)

\( (a+b)^5 = \binom{5}{0} a^5 + \binom{5}{1} a^4 b + \binom{5}{2} a^3 b^2 + \binom{5}{3} a^2 b^3 + \binom{5}{4} a b^4 + \binom{5}{5} b^5 \)

\( (a+b)^5 = 1 \cdot a^5 + 5 \cdot a^4 b + 10 \cdot a^3 b^2 + 10 \cdot a^2 b^3 + 5 \cdot a b^4 + 1 \cdot b^5 \)
Particular cases of Binomial Theorem

\[(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\]

If \(a = b = 1\)

\[n = 0 \quad 1 \quad = 2^0\]

\[n = 1 \quad 1 + 1 \quad = 2^1\]

\[n = 2 \quad 1 + 2 + 1 \quad = 2^2\]

\[n = 3 \quad 1 + 3 + 3 + 1 \quad = 2^3\]

\[n = 4 \quad 1 + 4 + 6 + 4 + 1 \quad = 2^4\]

If \(a = 10, \ b = 1\)

\[n = 0 \quad (10+1)^0 = 1 \cdot 10^0 \quad = 1\]

\[n = 1 \quad (10+1)^1 = 1 \cdot 10^1 + 1 \cdot 10^0 \quad = 11\]

\[n = 2 \quad (10+1)^2 = 1 \cdot 10^2 + 2 \cdot 10^1 + 1 \cdot 10^0 \quad = 121\]

\[n = 3 \quad (10+1)^3 = 1 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10^1 + 1 \cdot 10^0 \quad = 1331\]
The diagonals in Pascal’s Triangle

1, 1, 1, 1, 1, ...  
A constant sequence

1, 2, 3, 4, 5, 6, ...  
The sequence of natural numbers

The sequence of triangular numbers

The sequence of tetrahedral numbers

1, 5, 15, 35, 70, ...  
The sequence of 4-simplex numbers

Henri Poincaré, about algebraic topology
Fibonacci sequence in Pascal’s Triangle

\[ F_0 = 0 \quad F_1 = 1 \quad F_n = F_{n-1} + F_{n-2} \]

0, 1, 1, 2, 3, 5, 8, ….
Other properties of Pascal’s Triangle

### Divisibility

- **1 + 3 + 6 + 10 = 20**

### Hockey-stick identity

\[ \binom{n}{r} = \binom{n+1}{k+1} \]

\[ \sum_{r=k}^{n} \binom{n}{r} = \binom{n+1}{k+1} \]

\[ 2\binom{3}{2} + 3\binom{4}{2} + 5\binom{5}{2} = 6\binom{6}{3} \]
Is there some geometry in Pascal’s Triangle?

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<table>
<thead>
<tr>
<th>Number of points</th>
<th>Number of segments</th>
<th>Number of triangles</th>
<th>Number of quadrilaterals</th>
<th>Number of pentagons</th>
<th>Number of hexagons</th>
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</table>

A → B → P → N
M
Q → R → S → T → X → V
Y → Z → U

Nicoleta Babutiu
CEMC: Bringing Teachers Together Virtually | August 20, 2020
## Probabilities in Pascal's Triangle?

### Possible outcomes

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<td>2^2</td>
<td>2^3</td>
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### Total number of outcomes

- **n = number of toss**
- **X = number of heads**

\[
P(X=k) = \binom{n}{k} \frac{1}{2^n}
\]

\[
P(X=1) = \binom{3}{1} \frac{1}{2^3} = \frac{3}{8}
\]

<table>
<thead>
<tr>
<th>k</th>
<th>P(X=k)</th>
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<tbody>
<tr>
<td>0</td>
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<tr>
<td>2</td>
<td>1/4</td>
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<tr>
<td>3</td>
<td>1/8</td>
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*Pascal's Triangle*
Sierpinski’s Triangle

Wacław Sierpiński described the Sierpinski Triangle in 1915. It is a fractal with the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles.

However, similar patterns appear already in the 13th-century Cosmati mosaics in the cathedral of Anagni, Italy. 

https://www.pinterest.ca/leterrae/italy-tour-cosmatesque-pavements/
Constructing the Sierpinski Triangle

1. Shrinking and duplication

Step 1: Start with an equilateral triangle

Step 2: Shrink the triangle to 1/2 height and 1/2 width, make three copies

Step 3: Repeat step 2 with each of the smaller triangles
Constructing the Sierpinski Triangle

2. Removing triangles

Step 1: Start with an equilateral triangle

Step 2: Subdivide it into four smaller congruent equilateral triangles and remove the central triangle

Step 3: Repeat step 2 with each of the remaining smaller triangles infinitely
## Constructing the Sierpinski Triangle

### 3. Pascal's Triangle

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<td>n=6</td>
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<tr>
<td>n=7</td>
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**Pascal's Triangle**

**Pascal's Triangle (even/odd numbers)**
Constructing the Sierpinski triangle

**Definition**  If $a$ and $b$ are integers and $n > 0$, we write $a \equiv b \pmod{n}$ if and only if $n|(b - a)$. We read this as “$a$ is congruent to $b$ modulo $n$”.

**Examples:**

- $5 \equiv 2 \pmod{3}$, $9 \equiv 0 \pmod{3}$, $10 \equiv 1 \pmod{3}$
- $15 \equiv 0 \pmod{5}$, $9 \equiv 4 \pmod{5}$, $11 \equiv 1 \pmod{5}$
- $15 \equiv 1 \pmod{7}$, $9 \equiv 2 \pmod{7}$, $11 \equiv 4 \pmod{7}$

Quantum Pascal’s Triangle and Sierpinski’s carpet, 2017

Tom Bannink\*, Harry Buhrman\*\‡
Summary

• Pascal’s Triangle opens many different patterns. The more familiar students become with these patterns, the easier it will be for them to understand mathematics.

• Pascal’s triangle can be taught at different grade levels while showing the students that mathematics is all around us.

• The connections between Pascal’s triangle and geometry, probabilities, fractals and arts grant us the opportunity to teach difficult topics in mathematics by using patterns in a more engaging way

• What is math about?
Q&A

Thank you!

Merci!
References

https://en.wikipedia.org/wiki/Pascal%27s_simplex


https://www.youtube.com/watch?v=J0l1NuxUcpQ

https://en.wikipedia.org/wiki/Sierpi%C5%84ski_triangle#:~:text=The%20Sierpi%C5%84ski%20triangle%20(sometimes%20spelled,recursively%20into%20smaller%20equilateral%20triangles.

https://arxiv.org/abs/1708.07429
