



The CENTRE for EDUCATION  
in MATHEMATICS and COMPUTING  
*cemc.uwaterloo.ca*

## ***2023 Fryer Contest***

**Wednesday, April 5, 2023**  
(in North America and South America)

**Thursday, April 6, 2023**  
(outside of North America and South America)

*Solutions*

1. (a) Lonnie rests for 30 s between the 1st and 2nd sprints, the 2nd and 3rd sprints, and so on up to and including the 23rd and 24th sprints.  
Thus, Lonnie takes the 30 s rest 23 times.

- (b) Since Lonnie sprints at a constant speed of 8 m/s, then it takes Lonnie  $\frac{200 \text{ m}}{8 \text{ m/s}} = 25 \text{ s}$  to sprint 200 m.

Lonnie completes 24 such sprints, and so his time spent sprinting is  $24 \times 25 \text{ s} = 600 \text{ s}$ .

Lonnie also takes 23 rests, each of length 30 s, and so his time spent resting is  $23 \times 30 \text{ s} = 690 \text{ s}$ .

On Monday, Lonnie's total practice time is thus  $600 \text{ s} + 690 \text{ s} = 1290 \text{ s}$ .

- (c) *Solution 1*

On Tuesday, each of Lonnie's 240 m sprints takes  $\frac{240 \text{ m}}{8 \text{ m/s}} = 30 \text{ s}$ , and so Lonnie spends  $20 \times 30 \text{ s} = 600 \text{ s}$  sprinting.

Lonnie rests 19 times, and so he rests for a total of  $19 \times 30 \text{ s} = 570 \text{ s}$ .

On Tuesday, Lonnie's total practice time is thus  $600 \text{ s} + 570 \text{ s} = 1170 \text{ s}$ , and so Tuesday's practice takes  $1290 - 1170 = 120$  fewer seconds compared to Monday's practice.

*Solution 2*

On Monday, Lonnie sprints  $24 \times 200 \text{ m} = 4800 \text{ m}$ .

On Tuesday, Lonnie also sprints  $20 \times 240 \text{ m} = 4800 \text{ m}$ .

Since Lonnie sprints at the same constant speed on both days, then he spends the same amount of time sprinting on each of the two days.

Thus, the difference between the length of time that Lonnie practices on the two days is the difference between the time that he spends resting between sprints.

Lonnie rests 23 times on Monday, and he rests 19 times on Tuesday.

Since he rests 4 more times on Monday than he does on Tuesday, then Tuesday's practice takes  $4 \times 30 = 120$  fewer seconds compared to Monday's practice.

2. (a) The 5th row includes the integers 17, 19, 21, 23, and 25, and so the average of the integers in the 5th row is  $\frac{17 + 19 + 21 + 23 + 25}{5} = 21$ .

- (b) The row that has the integer 145 in the 1st position must immediately follow the row that has 144 in the last position.

Since  $12^2 = 144$ , then the integer 144 is in the last position of the 12th row, and so 145 is in the first position of the 13th row.

- (c) Since  $40^2 = 1600$ , then the integer in the last position (the 40th position) of the 40th row is 1600.

When moving from right to left along each row, the integers decrease by 2, and so the integer in the 39th position of the 40th row is  $1600 - 2 = 1598$ .

- (d) *Solution 1*

Moving from left to right along a row, the integers increase by a constant (namely 2), and so the average of the integers in a row is equal to the average of the integer in the first position of the row and the integer in the last position of the row. Can you see why this is true?

Since  $15^2 = 225$ , then the integer in the last position of the 15th row is 225, and so the integer in the first position of the 16th row is 226.

Since  $16^2 = 256$ , then the integer in the last position of the 16th row is 256.

Thus, the average of the integers in the 16th row is  $\frac{226 + 256}{2} = 241$ , and so  $r = 16$ .

*Solution 2*

Since  $15^2 = 225$ , then each of the entries in the first 15 rows is at most 225.

This means that the average of the entries in each row up to and including the 15th row must be at most 225.

Since  $16^2 = 256$ , then each of the entries in the rows after the 16th row is greater than 256. This means that the average of the entries in each row after the 16th must be greater than 256.

This means  $r$  must be greater than 15 and must be smaller than 17. In other words,  $r = 16$ .

We can check that the entries in row 16 are

$$226, 228, 230, 232, 234, 236, 238, 240, 242, 244, 246, 248, 250, 252, 254, 256$$

and the average of these integers is indeed 241.

3. (a) The five-digit positive integer  $4B5B2$  is divisible by 3 exactly when the sum of its digits,  $4 + B + 5 + B + 2 = 2B + 11$  is divisible by 3.

Checking the possible values of  $B$ ,

Value of $B$	1	2	3	4	5	6	7	8	9
Value of $2B + 11$	13	15	17	19	21	23	25	27	29

we get that  $2B + 11$  is divisible by 3 when  $B = 2$ ,  $B = 5$  and  $B = 8$ .

- (b) *Solution 1*

Since  $ABABA$  is not divisible by 3, then  $A + B + A + B + A = 3A + 2B$  is not divisible by 3.

Since  $3A$  is divisible by 3 for all possible values of the digit  $A$ , then if  $2B$  were also divisible by 3 (that is, if  $B = 3, 6, 9$ ), it would be the case that  $3A + 2B$  is divisible by 3.

Since  $3A + 2B$  is not divisible by 3, then  $2B$  cannot be divisible by 3 and so the possible values of  $B$  are 1, 2, 4, 5, 7, 8.

Since  $ABABA$  is divisible by 4, then the two-digit positive integer  $BA$  is divisible by 4.

For each of the possible values of  $B$ , namely  $B = 1, 2, 4, 5, 7, 8$ , we determine the values of  $A$  for which  $BA$  is divisible by 4.

For example when  $B = 1$ , the two-digit positive integer  $1A$  is divisible by 4 exactly when  $A = 2$  or  $A = 6$ .

In the table below, we determine the remaining pairs  $A$  and  $B$  that are possible.

$B$	$A$	$(A, B)$
1	2, 6	(2, 1), (6, 1)
2	4, 8	(4, 2), (8, 2)
4	4, 8	(4, 4), (8, 4)
5	2, 6	(2, 5), (6, 5)
7	2, 6	(2, 7), (6, 7)
8	4, 8	(4, 8), (8, 8)

Thus, there are 12 different pairs of non-zero digits  $A$  and  $B$  that are possible.

*Solution 2*

Since  $ABABA$  is divisible by 4, then it is also divisible by 2 and thus even.

Since  $ABABA$  is even, then the ones digit is even, and so the possible values of  $A$  are 2, 4, 6, 8.

Since  $ABABA$  is divisible by 4, then the two-digit positive integer  $BA$  is divisible by 4.

For each of the possible values of  $A$ , namely  $A = 2, 4, 6, 8$ , we determine the values of  $B$  for which  $BA$  is divisible by 4.

For example when  $A = 2$ , the two-digit positive integer  $B2$  is divisible by 4 exactly when  $B = 1, 3, 5, 7$ , and 9.

When  $A = 4$ , the two-digit positive integer  $B4$  is divisible by 4 exactly when  $B = 2, 4, 6$ , and 8.

When  $A = 6$ , the two-digit positive integer  $B6$  is divisible by 4 exactly when  $B = 1, 3, 5, 7$ , and 9.

When  $A = 8$ , the two-digit positive integer  $B8$  is divisible by 4 exactly when  $B = 2, 4, 6$ , and 8.

Finally, we consider the fact that  $ABABA$  is not divisible by 3.

As was shown in Solution 1, the possible values of  $B$  are 1, 2, 4, 5, 7, 8 ( $B \neq 3, 6, 9$ ).

Combining this information with the previous values of  $A$  and  $B$ , we get that the possible pairs of non-zero digits  $(A, B)$  are (2, 1), (2, 5), (2, 7), (4, 2), (4, 4), (4, 8), (6, 1), (6, 5), (6, 7), (8, 2), (8, 4), and (8, 8).

Thus, there are 12 different pairs of non-zero digits  $A$  and  $B$  that are possible.

- (c) If  $t = ACA2 \times BAC$  is divisible by 15, then  $t$  is divisible by both 5 and 3.

An integer is divisible by 5 exactly when its ones digit is 0 or 5, and so  $ACA2$  is not divisible by 5.

Since  $t = ACA2 \times BAC$  is divisible by 5 and  $ACA2$  is not, then  $BAC$  must be divisible by 5, which means that  $C = 5$  (since  $C$  is a non-zero digit).

Substituting  $C = 5$ , we get  $t = A5A2 \times BA5$ .

Since  $t$  is divisible by 3 and 3 is a prime number, then at least one of  $A5A2$  or  $BA5$  is divisible by 3, and so  $A + 5 + A + 2 = 2A + 7$  is divisible by 3, or  $B + A + 5$  is divisible by 3, or both are divisible by 3.

Since  $t$  is not divisible by 12, but  $t$  is divisible by 3, then  $t$  is not divisible by 4.

The three-digit integer  $BA5$  is not divisible by 2 (and thus not divisible by 4) for all possible values of  $A$ , and so the four-digit integer  $A5A2$  must not be divisible by 4.

The four-digit integer  $A5A2$  is divisible by 4 when the two-digit integer  $A2$  is divisible by 4, or when  $A = 1, 3, 5, 7$ , or 9, and so the possible values of  $A$  are 2, 4, 6, 8.

Finally, we return to the requirement that  $t = A5A2 \times BA5$  is divisible by 3, meaning that at least one of  $2A + 7$  or  $B + A + 5$  is divisible by 3.

When  $A = 2$ ,  $2A + 7 = 11$  which is not divisible by 3 and so  $B + A + 5 = B + 7$  must be divisible by 3.  $B + 7$  is divisible by 3 exactly when  $B = 2, 5$ , or 8, and so there are 3 triples  $A, B, C$  that are possible in this case.

When  $A = 4$ ,  $2A + 7 = 15$  which is divisible by 3, which means that  $B$  can be equal to any non-zero digit, and so there are 9 triples  $A, B, C$  that are possible in this case.

When  $A = 6$ ,  $2A + 7 = 19$  which is not divisible by 3 and so  $B + A + 5 = B + 11$  must be divisible by 3.  $B + 11$  is divisible by 3 exactly when  $B = 1, 4$ , or 7, and so there are 3 triples  $A, B, C$  that are possible in this case.

When  $A = 8$ ,  $2A + 7 = 23$  which is not divisible by 3 and so  $B + A + 5 = B + 13$  must

be divisible by 3.  $B + 13$  is divisible by 3 exactly when  $B = 2, 5$  or  $8$ , and so there are 3 triples  $A, B, C$  that are possible in this case.

Therefore, there are  $3 + 9 + 3 + 3 = 18$  different triples of non-zero digits  $A, B, C$  that are possible.

4. (a) Computer 1 is an odd-numbered computer, and so each cord connecting Computer 1 to another odd-numbered computer is red.

Thus, there is a route from Computer 1 to each of the odd-numbered computers from 3 to 49, inclusive, that uses only red cords.

Each cord between an odd-numbered computer and an even-numbered computer is blue. Computer 1 is an odd-numbered computer and so every possible route from Computer 1 to an even-numbered computer must use at least one blue cord.

Thus there is no route from Computer 1 to any even-numbered computer that uses only red cords.

There are 24 odd numbers between 2 and 50, and so there are 24 possible values for  $n$ .

- (b) Two integers have *different parity* if one integer is even and the other is odd.

Two integers have the *same parity* if they are both even or if they are both odd.

There are two cases to consider:  $A$  and  $B$  have different parity, or they have the same parity.

If  $A$  and  $B$  have different parity, then the cord between Computer  $A$  and Computer  $B$  is blue, and so there is a route between them that uses only blue cords.

If  $A$  and  $B$  have the same parity, then choose a number  $C$  that has different parity than that of  $A$  and  $B$ .

The cord between Computer  $A$  and Computer  $C$  is blue, and the cord between Computer  $C$  and Computer  $B$  is blue, and so the route from Computer  $A$  to Computer  $C$  to Computer  $B$  uses only blue cords.

Thus, for every pair of distinct computers, Computer  $A$  and Computer  $B$ , there is always a route between them that uses only blue cords.

- (c) If the cord between Computer 13 and Computer 14 is yellow, then there is a route between them that uses only yellow cords, so assume that the cord between them is green.

Since there is no route connecting Computer 1 to Computer 50 that uses only green cords, then the cord between Computer 1 and Computer 50 must be yellow.

Further, since there is no route connecting Computer 1 to Computer 50 that uses only green cords, then at least one of the following must be true:

- (i) the cord between Computer 1 and Computer 13 is yellow, or
- (ii) the cord between Computer 13 and Computer 50 is yellow,

otherwise the route from Computer 1 to Computer 13 to Computer 50 uses only green cords.

Similarly, since there is no route connecting Computer 1 to Computer 50 that uses only green cords, then at least one of the following must be true:

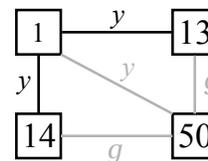
- (iii) the cord between Computer 1 and Computer 14 is yellow, or
- (iv) the cord between Computer 14 and Computer 50 is yellow,

otherwise the route from Computer 1 to Computer 14 to Computer 50 uses only green cords.

Since at least one of (i) or (ii) must be true, and at least one of (iii) or (iv) must be true, then there are 4 cases to consider, as follows.

Case A: (i) and (iii) are true

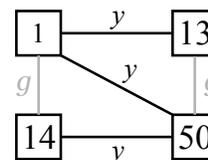
In this case, the cord between Computer 1 and Computer 13 is yellow, and the cord between Computer 1 and Computer 14 is yellow, and so the route from Computer 13 to Computer 1 to Computer 14 uses only yellow cords.



Case B: (i) and (iv) are true

In this case, the cord between Computer 1 and Computer 13 is yellow, and the cord between Computer 14 and Computer 50 is yellow.

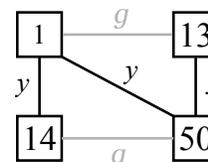
Recall that the cord between Computer 1 and Computer 50 is also yellow, and so the route from Computer 13 to Computer 1 to Computer 50 to Computer 14 uses only yellow cords.



Case C: (ii) and (iii) are true

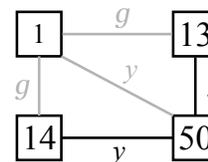
In this case, the cord between Computer 13 and Computer 50 is yellow, and the cord between Computer 1 and Computer 14 is yellow.

Since the cord between Computer 1 and Computer 50 is also yellow, then the route from Computer 13 to Computer 50 to Computer 1 to Computer 14 uses only yellow cords.



Case D: (ii) and (iv) are true

In this case, the cord between Computer 13 and Computer 50 is yellow, and the cord between Computer 14 and Computer 50 is yellow, and so the route from Computer 13 to Computer 50 to Computer 14 uses only yellow cords.



Thus, if there is no route that connects Computer 1 to Computer 50 that uses only green cords, then there is always a route between Computer 13 and Computer 14 that uses only yellow cords.



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## ***2022 Fryer Contest***

**Tuesday, April 12, 2022**  
(in North America and South America)

**Wednesday, April 13, 2022**  
(outside of North America and South America)

*Solutions*

1. (a) Each hit adds 7 points to Shane's score, and so 4 hits adds  $7 \times 4 = 28$  points to his score. For each miss, 3 points are subtracted from Shane's score, and so 2 misses subtracts  $3 \times 2 = 6$  points from his score.

A player's score begins at 0, and so after these 6 throws, Shane's score is  $28 - 6 = 22$ .

- (b) *Solution 1*

After exactly  $h$  hits and 6 misses, Susan's score is given by the expression  $7 \times h - 3 \times 6$  or  $7h - 18$ .

After these throws, Susan's score is 59, and so  $7h - 18 = 59$ .

Solving this equation, we get  $7h = 59 + 18$  or  $7h = 77$ , and so  $h = 11$ .

*Solution 2*

Susan's 6 misses decrease her score by  $3 \times 6 = 18$  points.

Since her score is 59, then Susan must have scored  $59 + 18 = 77$  points from hits.

Since each hit is worth 7 points, then the value of  $h$  is  $77 \div 7 = 11$ .

- (c) *Solution 1*

We begin by making an initial guess at the value of  $m$ , and then systematically adjust the value upward or downward as needed.

If we begin with  $m = 3$  for example, then the number of hits is  $20 - 3 = 17$  and Souresh's score is  $7 \times 17 - 3 \times 3 = 119 - 9 = 110$ .

Since this score is greater than 105, then the value of  $m$  is greater than 3 (more misses means fewer hits and a lower score).

When  $m = 4$ , the number of hits is  $20 - 4 = 16$  and Souresh's score is  $7 \times 16 - 3 \times 4 = 112 - 12 = 100$ .

Since this score is greater than 85 and less than 105, then 4 is a possible value of  $m$ .

When  $m = 5$ , the number of hits is  $20 - 5 = 15$  and Souresh's score is  $7 \times 15 - 3 \times 5 = 105 - 15 = 90$ .

This score is also greater than 85 and less than 105, and so 5 is a possible value of  $m$ .

When  $m = 6$ , the number of hits is 14 and Souresh's score is  $7 \times 14 - 3 \times 6 = 98 - 18 = 80$ .

This score is less than 85 and so 6 is not a possible value of  $m$ .

Continuing to increase the number of misses will further decrease Souresh's score, and thus the only possible values are  $m = 4$  and  $m = 5$ .

*Solution 2*

Since Souresh makes 20 throws and  $m$  of these throws are misses, then the remaining  $20 - m$  throws are hits.

After exactly  $20 - m$  hits and  $m$  misses, Souresh's score is given by the expression  $7(20 - m) - 3m$  or  $140 - 10m$ .

Since Souresh's score ( $140 - 10m$ ) is greater than 85, then  $10m$  must be less than 55 (note that  $140 - 85 = 55$ ) and so  $m < \frac{55}{10}$ .

Since  $\frac{55}{10} = 5\frac{1}{2}$  and  $m$  is a positive integer, then  $m \leq 5$ .

Since Souresh's score is less than 105, then  $10m$  must be greater than 35 (note that  $140 - 105 = 35$ ) and so  $m > \frac{35}{10}$ .

Since  $\frac{35}{10} = 3\frac{1}{2}$  and  $m$  is a positive integer, then  $m \geq 4$ .

Thus,  $m$  is a positive integer and  $4 \leq m \leq 5$ , and so  $m = 4$  or  $m = 5$ .

(We can check that when  $m = 4$ , Souresh's score is  $7 \times 16 - 3 \times 4 = 112 - 12 = 100$ , and when  $m = 5$ , Souresh's score is  $7 \times 15 - 3 \times 5 = 105 - 15 = 90$ .)

2. (a) *Solution 1*

The area of  $ABGH$  is equal to the sum of the areas of  $ABCD$  and  $EFGH$  minus the area of overlap  $EFCD$  since it is counted twice in this sum.

Thus, the area of  $ABGH$  is  $(13 \text{ cm}^2) + (13 \text{ cm}^2) - (5 \text{ cm}^2) = 21 \text{ cm}^2$ .

*Solution 2*

The area of  $ABCD$  is  $13 \text{ cm}^2$  and is equal to the sum of the areas of  $ABFE$  and  $EFCD$ . Since the area of  $EFCD$  is  $5 \text{ cm}^2$ , then the area of  $ABFE$  is  $(13 \text{ cm}^2) - (5 \text{ cm}^2) = 8 \text{ cm}^2$ .

The area of  $ABGH$  is equal to the sum of the areas of  $ABFE$  and  $EFGH$  or  $(8 \text{ cm}^2) + (13 \text{ cm}^2) = 21 \text{ cm}^2$ .

(b) Let the area of the overlapped region,  $\triangle KLN$ , be  $x \text{ cm}^2$ .

The area of  $\triangle KLN$  is equal to half of the area of  $\triangle JKL$ , and so the area of  $\triangle JKN$  is also  $x \text{ cm}^2$ .

Since  $\triangle JKL$  and  $\triangle MLK$  are identical, then the area of  $\triangle MLN$  is also  $x \text{ cm}^2$ .

Thus, the area of the figure  $JKLMN$  is  $3x \text{ cm}^2$ , and so  $3x = 48$  or  $x = 16$ .

Since  $\triangle JKL$  is right-angled at  $K$ , then its area is given by  $\frac{1}{2}(JK)(KL)$ .

Since the area of  $\triangle JKL$  is  $2x \text{ cm}^2 = 32 \text{ cm}^2$ , then  $\frac{1}{2}(JK)(KL) = 32 \text{ cm}^2$  or  $\frac{1}{2}(6 \text{ cm})(KL) = 32 \text{ cm}^2$ , and so  $KL = \frac{32}{3} \text{ cm}$ .

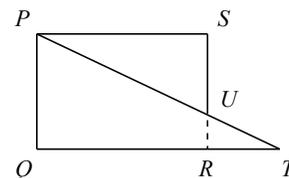
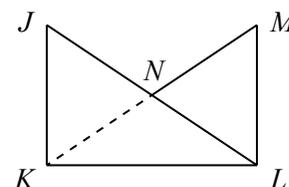
(c) We use the notation  $|\triangle URT|$  to denote the area of  $\triangle URT$ , and similar notation for other areas.

Since  $|PQRS| + |\triangle URT| = |PQTUS|$ , then

$$|\triangle URT| = |PQTUS| - |PQRS| = 117 \text{ cm}^2 - 108 \text{ cm}^2 = 9 \text{ cm}^2.$$

Since  $|PQRU| + |\triangle URT| = |\triangle PQT|$ , then

$$|PQRU| = |\triangle PQT| - |\triangle URT| = 81 \text{ cm}^2 - 9 \text{ cm}^2 = 72 \text{ cm}^2.$$

3. (a) The prime factorization of  $675 = 3^3 \times 5^2$ , and so 675 has  $4 \times 3 = 12$  positive factors.(b) The positive integer  $n$  has a total of  $4 + 14 = 18$  positive factors.

Since  $n$  has the positive factors  $9 = 3^2$ ,  $11$ ,  $15 = 3 \times 5$ , and  $25 = 5^2$ , then the prime factorization of  $n$  must include at least 2 factors of 3, at least 2 factors of 5, and at least 1 factor of 11.

In other words,  $n$  must be divisible by  $3^2 \times 5^2 \times 11$ .

Suppose  $n = 3^2 \times 5^2 \times 11$ . Then  $n$  has  $3 \times 3 \times 2 = 18$  positive factors, as required.

If  $n$  contained additional factors, then it would have more than 18 positive factors.

Thus,  $n = 3^2 \times 5^2 \times 11 = 2475$ .

(c) Suppose that  $m$  is a positive integer less than 500 that has exactly  $2 + 10 = 12$  positive factors.

Since  $m$  has the positive factors 2 and  $9 = 3^2$ , then the prime factorization of  $m$  must include at least 1 factor of 2, and at least 2 factors of 3.

In other words,  $m$  must be divisible by  $2 \times 3^2$ .

To begin, suppose that  $m$  has exactly 2 distinct prime factors.

That is, suppose that  $m = 2^a \times 3^b$  where  $a$  and  $b$  are integers with  $a \geq 1$  and  $b \geq 2$ .

In this case,  $m$  has  $(a + 1)(b + 1) = 12$  positive factors.

Since  $a \geq 1$  and  $b \geq 2$ , then  $a + 1 \geq 2$  and  $b + 1 \geq 3$ .

Using these restrictions, there are exactly three possibilities for which  $(a + 1)(b + 1) = 12$ . These are

$$a + 1 = 2 \text{ and } b + 1 = 6, \text{ which gives } a = 1 \text{ and } b = 5$$

$$a + 1 = 3 \text{ and } b + 1 = 4, \text{ which gives } a = 2 \text{ and } b = 3$$

$$a + 1 = 4 \text{ and } b + 1 = 3, \text{ which gives } a = 3 \text{ and } b = 2$$

If  $a = 1$  and  $b = 5$ , then  $m = 2 \times 3^5 = 486$ .

If  $a = 2$  and  $b = 3$ , then  $m = 2^2 \times 3^3 = 108$ .

If  $a = 3$  and  $b = 2$ , then  $m = 2^3 \times 3^2 = 72$ .

Since each of these values is less than 500, then there are 3 positive integers that satisfy the given conditions, in this case.

Next, suppose that  $m$  has exactly 3 distinct prime factors.

That is, suppose that  $m = 2^a \times 3^b \times p^c$  where  $p$  is a prime number not equal to 2 or 3, and  $a$ ,  $b$  and  $c$  are integers with  $a \geq 1$ ,  $b \geq 2$  and  $c \geq 1$ .

If  $a = 1$ ,  $b = 2$  and  $c = 1$  (the minimum values possible for  $a, b, c$ ), then  $m = 2 \times 3^2 \times p$ .

In this case,  $m$  has  $2 \times 3 \times 2 = 12$  positive factors, as required.

Increasing  $a$ ,  $b$  or  $c$  increases the number of positive factors, and thus  $a = 1$ ,  $b = 2$  and  $c = 1$  is the only possibility for which  $m$  has 12 positive factors and 3 distinct prime factors.

If  $a = 1$ ,  $b = 2$  and  $c = 1$ , then  $m = 2 \times 3^2 \times p = 18p$ .

For which prime numbers  $p > 3$  is  $18p$  less than 500?

Since  $18p < 500$ , then  $p < \frac{500}{18}$  and so  $p \leq 27$ .

The prime numbers in this range are 5, 7, 11, 13, 17, 19, and 23, which give 7 positive integers that satisfy the given conditions, in this case.

Finally, suppose that  $m$  has exactly 4 distinct prime factors.

That is, suppose that  $m = 2^a \times 3^b \times p^c \times q^d$  where  $p$  and  $q$  are different prime numbers not equal to 2 or 3, and  $a$ ,  $b$ ,  $c$ ,  $d$  are integers with  $a \geq 1$ ,  $b \geq 2$ ,  $c \geq 1$ , and  $d \geq 1$ .

If  $a = 1$ ,  $b = 2$ ,  $c = 1$ , and  $d = 1$  (the minimum values possible for  $a, b, c, d$ ), then  $m$  has  $2 \times 3 \times 2 \times 2 = 24$  positive factors, which is a contradiction.

Increasing  $a$ ,  $b$ ,  $c$ , or  $d$  or increasing the number of distinct prime factors, increases the number of positive factors, and thus there are no possibilities for which  $m$  has 12 positive factors and 4 or more distinct prime factors.

Thus, the number of positive integers less than 500 that have the factors 2 and 9 and exactly ten other positive factors is  $3 + 7 = 10$ .

4. (a) Since one of the jars contains 0 beans, then all beans must be removed from the jar that contains 40 beans.

On Franco's turns, the number of beans that he removes is 1, 3, 4, 1, 3, 4,  $\dots$ , and so on. After Franco's 5 turns, he has removed a total of  $1 + 3 + 4 + 1 + 3 = 12$  beans.

On Sarah's turns, the number of beans that she removes is 2, 5, 2, 5, 2, 5,  $\dots$ , and so on. After Sarah's 5 turns, she has removed a total of  $2 + 5 + 2 + 5 + 2 = 16$  beans.

After a total of 10 turns, the total number of beans removed by Franco and Sarah is less than 40, and so we know that each of the two players was able to remove the required number of beans on each of their 5 turns.

After a total of 10 turns, the total number of beans left in the two jars is  $40 - (12 + 16) = 40 - 28 = 12$ .

- (b) Since one of the jars contains 0 beans, then all beans must be removed from the jar that contains 384 beans.

Franco repeatedly removes 1, 3 and 4 beans, which is a cycle of length 3.

Sarah repeatedly removes 2 and 5 beans, which is a cycle of length 2.

Since the lowest common multiple of 3 and 2 is 6, then after each player has had 6 turns (12 turns in total), they will each be back at the start of their cycle.

After the first 12 turns (6 turns for each player), Franco removes

$$1 + 3 + 4 + 1 + 3 + 4 = 2(1 + 3 + 4) = 16 \text{ beans,}$$

and Sarah removes

$$2 + 5 + 2 + 5 + 2 + 5 = 3(2 + 5) = 21 \text{ beans.}$$

On each of the following groups of 12 turns, Franco will again remove 16 beans and Sarah will again remove 21 beans.

From the start of the game, together they will remove  $16 + 21 = 37$  beans every 12 turns. After 10 such groups of turns (a total of  $10 \times 12 = 120$  turns), the total number of beans removed is  $10 \times 37 = 370$ , and so  $384 - 370 = 14$  beans are left in the jar.

After these 120 turns, Franco and Sarah are at the beginning of their sequences and it is Franco's turn.

On the 121<sup>st</sup> turn, Franco removes 1 bean, and so  $14 - 1 = 13$  beans remain.

On the 122<sup>nd</sup> turn, Sarah removes 2 beans, and so  $13 - 2 = 11$  beans remain.

On the 123<sup>rd</sup> turn, Franco removes 3 beans, and so  $11 - 3 = 8$  beans remain.

On the 124<sup>th</sup> turn, Sarah removes 5 beans, and so  $8 - 5 = 3$  beans remain.

The next turn is Franco's and he is not able to remove the required number of beans, 4, and thus Franco loses.

Therefore, Sarah wins after exactly  $n = 124$  turns.

- (c) We use the notation  $Tn$  to represent turn  $n$ , and  $(x, y)$  to represent the distribution of  $x$  beans in one jar and  $y$  in the other jar.

The game begins with the distribution  $(17, 6)$ , which is a total of  $17 + 6 = 23$  beans.

After  $T4$  (2 turns each), Franco has removed  $1 + 3 = 4$  beans and Sarah has removed  $2 + 5 = 7$  beans.

Thus after  $T4$ , there are a total of  $23 - (4 + 7) = 12$  beans remaining in the jars.

With 12 beans remaining, at least one of the jars must contain 6 or more beans, since if both jars contained fewer than 6 beans, then the total number of beans would be at most  $5 + 5 = 10$ .

Since at least one of the jars contains 6 or more beans, then on  $T5$ , Franco is able to remove 4 beans leaving a total of  $12 - 4 = 8$  beans.

With 8 beans remaining, at least one of the jars must contain 4 or more beans, and so on  $T6$ , Sarah is able to remove 2 beans, and on  $T7$ , Franco is able to remove 1 bean.

After  $T7$  (4 turns for Franco and 3 for Sarah), the total number of beans remaining in the jars is  $8 - (2 + 1) = 5$ .

Sarah must remove 5 beans on  $T8$  (Sarah's 4<sup>th</sup> turn).

If the distribution of beans is  $(5, 0)$ , then Sarah will remove the 5 beans and win on  $T8$ .

However, if the distribution is  $(4, 1)$  or  $(3, 2)$ , then Sarah will not be able to remove 5 beans and so Franco will win.

We summarize the solution to this point:

- It is not possible for either player to win on  $T1$  through  $T6$
- After  $T7$ , the total number of beans remaining in the jars is 5
- On  $T8$ , it is Sarah's turn and she must remove 5 beans

- After T7, Franco wins if he leaves distributions of  $(4, 1)$  or  $(3, 2)$
- After T7, Franco will lose if he leaves a distribution of  $(5, 0)$  (Sarah will win on T8)

Thus, Franco has a winning strategy if he can ensure that the final 5 beans are not all in one jar, otherwise Sarah has a winning strategy.

Franco has the winning strategy.

We summarize his strategy and then explain why this strategy guarantees that Franco will always win.

Franco's strategy:

- Remove 1 bean from the jar containing 17 beans on T1
- Remove 3 beans from the jar containing the fewest number of beans on T3
- Remove beans from the jar containing the greatest number of beans on T5 and T7

On T1, Franco removes 1 bean from the jar containing 17 beans, and so the distribution is  $(16, 6)$ .

On T2, Sarah removes 2 beans from one of the jars, and so the distribution is  $(14, 6)$  or  $(16, 4)$ .

On T3, Franco removes 3 beans from the jar containing the fewest number of beans.

Thus after exactly 3 turns, the jars contain  $(14, 3)$ , or they contain  $(16, 1)$ .

Through T4 to T7 inclusive,  $5 + 4 + 2 + 1 = 12$  more beans are removed from the jars.

Thus in each of the above two cases, the jar containing the greatest number of beans (14 and 16) can not be emptied.

Can Sarah empty a jar that contains 3 beans or that contains 1 bean without Franco removing any beans from these jars?

Since Sarah can remove 2 or 5 beans on each of her turns, she will be unable to remove exactly 3 beans and she will similarly be unable to remove exactly 1 bean.

This means that for each of the above two cases, Sarah will not be able to empty either of the jars.

Therefore, after exactly 7 turns the jars will contain  $(2, 3)$ , or  $(4, 1)$ , and thus Sarah will lose since she will not be able to remove 5 beans on T8.

- (d) There are a total of  $2023 + 2022 = 4045$  beans at the start of the game.

As was shown in part (b), 37 beans are removed in each successive group of 12 turns beginning from the start of the game.

After 109 such groups of turns (a total of  $109 \times 12 = 1308$  turns), the total number of beans removed is  $109 \times 37 = 4033$ , and so  $4045 - 4033 = 12$  beans are left in the jars.

After these 1308 turns, Franco and Sarah are at the beginning of their sequences and it is Franco's turn.

With 12 beans remaining, at least one of the jars must still contain 6 or more beans, and so Franco and Sarah were able to remove their required number of beans on each of the first 1308 turns.

From this point in the game, the winner is determined by how the 12 beans are distributed between the two jars.

For example, assume that the distribution of the final 12 beans is  $(12, 0)$ .

Franco removes 1, Sarah removes 2, Franco removes 3, Sarah removes 5, and then there is  $12 - (1 + 2 + 3 + 5) = 1$  bean remaining.

On the next turn, Franco is unable to remove 4 beans and so in this case Sarah wins.

Assume that the distribution of the final 12 beans is  $(11, 1)$ .

Franco removes 1, and so the jars contain  $(10, 1)$  or  $(11, 0)$ .

As we saw in the previous case, if the distribution is  $(11, 0)$  at this point in the game, then Sarah wins.

So we assume that Franco removes the 1 bean from the jar containing 11 beans, so that the jars contain  $(10, 1)$ .

In each of the next 3 turns, beans must be removed from the jar with 10 beans: Sarah removes 2, Franco removes 3, Sarah removes 5, and then there are  $10 - (2 + 3 + 5) = 0$  beans remaining in that jar and still 1 bean in the other jar.

On the next turn, Franco is unable to take 4 beans and so Sarah wins.

There are other ways that the final 12 beans can be distributed between the two jars, and for some of these distributions, Sarah is guaranteed to win.

However, to describe a winning strategy for Sarah concisely, we demonstrate how Sarah can force the distribution of the final 12 beans to be one of the two cases above, thus guaranteeing her win.

Next, we demonstrate how Sarah is able to force the distribution of the final 12 beans to be  $(12, 0)$  or  $(11, 1)$ .

Sarah's winning strategy is to remove all beans from one of the jars, or to remove all but 1 bean from one of the jars.

Calling this jar the *target* jar, her strategy is as follows:

- If the target jar contains 5 or more beans, Sarah removes beans from the target jar on each of her turns
- If the target jar contains 2, 3 or 4 beans, Sarah removes 2 beans from the target jar (when it's her turn to remove 2), and removes 5 beans from the other jar (when it's her turn to remove 5)
- Once the target jar contains 0 beans or 1 bean, Sarah removes beans from the other jar on each of her remaining turns

Finally, we explain why Sarah is able to perform this strategy.

In each successive group of 12 turns (6 turns for Sarah), Sarah removes  $3(2 + 5) = 21$  beans.

Thus in the first 109 groups of 12 turns (1308 turns), Sarah removes  $109 \times 21 = 2289$  beans.

Since the jars originally contain 2022 and 2023 beans, Sarah is able to remove enough beans so that the target jar contains 0 beans or 1 bean.

After the 1308 turns, the other jar contains 12 or 11 beans, which tells us that Sarah will be able to remove the required number of beans on each of her turns.

(We note that Franco may also remove beans from the target jar, however this is okay as it has no effect on Sarah's strategy.)

Summarizing, Sarah has a strategy that will ensure that the distribution of the final 12 beans is either  $(12, 0)$  or  $(11, 1)$ .

Given that the final 12 beans are distributed in one of these two ways, Sarah is guaranteed to win this game, and thus Sarah has a winning strategy.

It is worth noting that the player who has the winning strategy in this game, Sarah, is unique, however the argument demonstrating her guaranteed win, is not unique. For example, Sarah is also guaranteed to win if the distribution of the final 12 beans is  $(10, 2)$  or  $(9, 3)$ , and thus this could have also be used in the description of Sarah's winning strategy.



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## ***2021 Fryer Contest***

**April 2021**

(in North America and South America)

**April 2021**

(outside of North America and South America)

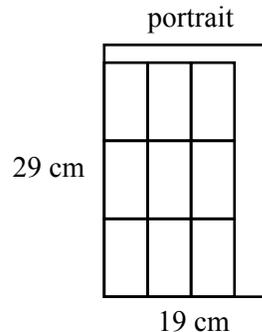
*Solutions*

1. (a) Each business card has dimensions  $5\text{ cm} \times 9\text{ cm}$  and thus has area  $5\text{ cm} \times 9\text{ cm} = 45\text{ cm}^2$ .  
 (b) Each single page with dimensions  $20\text{ cm} \times 27\text{ cm}$  has area  $20\text{ cm} \times 27\text{ cm} = 540\text{ cm}^2$ .  
 Each business card has area  $45\text{ cm}^2$ .

The entire page is used with no waste and so the number of business cards that can be printed without overlap is  $\frac{540\text{ cm}^2}{45\text{ cm}^2} = 12$ .

- (c) We begin by first considering the portrait page layout.

The width of each card is  $5\text{ cm}$  and the width of the page is  $19\text{ cm}$ . Since 3 adjacent cards give a combined width of  $15\text{ cm}$  (less than  $19\text{ cm}$ ) and 4 adjacent cards give a combined width of  $20\text{ cm}$  (greater than  $19\text{ cm}$ ), then the greatest number of adjacent cards that can be printed horizontally across the page in this layout is 3.

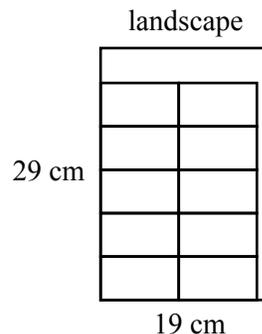


The height of each card is  $9\text{ cm}$  and the height of the page is  $29\text{ cm}$ . Since 3 adjacent cards give a combined height of  $27\text{ cm}$  (less than  $29\text{ cm}$ ) and 4 adjacent cards give a combined height of  $36\text{ cm}$  (greater than  $29\text{ cm}$ ), then the greatest number of adjacent cards that can be printed vertically on the page in this layout is 3.

Thus the portrait page layout allows a maximum of  $3 \times 3 = 9$  business cards to be printed on a single page, as shown in the diagram above.

Next, we consider the landscape page layout.

The width of each card is  $9\text{ cm}$  and the width of the page is  $19\text{ cm}$ . Since 2 adjacent cards give a combined width of  $18\text{ cm}$  (less than  $19\text{ cm}$ ) and 3 adjacent cards give a combined width of  $27\text{ cm}$  (greater than  $19\text{ cm}$ ), then the greatest number of adjacent cards that can be printed horizontally across the page in this layout is 2.



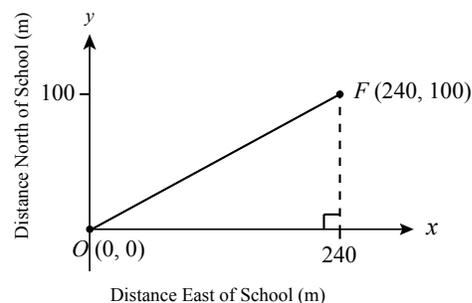
The height of each card is  $5\text{ cm}$  and the height of the page is  $29\text{ cm}$ . Since 5 adjacent cards give a combined height of  $25\text{ cm}$  (less than  $29\text{ cm}$ ) and 6 adjacent cards give a combined height of  $30\text{ cm}$  (greater than  $29\text{ cm}$ ), then the greatest number of adjacent cards that can be printed vertically on the page in this layout is 5.

Thus the landscape page layout allows a maximum of  $2 \times 5 = 10$  business cards to be printed on a single page, as shown.

Therefore, the landscape page layout allows the greatest number of business cards to be printed on a single page.

2. Throughout the solution to this problem, all coordinates represent lengths in metres.

- (a) Let the point  $A$  be positioned at  $(240, 0)$ . Then  $OA$  represents a horizontal distance of 240 m,  $AF$  represents a vertical distance of 100 m, and  $\triangle OAF$  is right-angled at  $A$ , as shown. By the Pythagorean Theorem,  $FO^2 = OA^2 + AF^2$  or  $FO^2 = (240 \text{ m})^2 + (100 \text{ m})^2 = 67\,600 \text{ m}^2$  and so  $FO = \sqrt{67\,600 \text{ m}^2} = 260 \text{ m}$ .

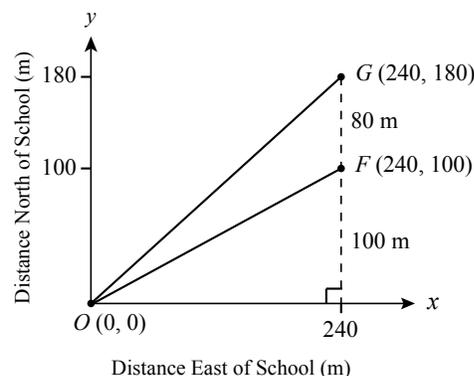


- (b) On Monday, Franklin walks from school straight home (a distance of 260 m) at a constant speed of 80 m/min. Since time equals distance divided by speed, it takes Franklin  $\frac{260 \text{ m}}{80 \text{ m/min}} = \frac{13}{4}$  (or 3.25) minutes to walk home on Monday.

- (c) Points  $F$  and  $G$  have the same  $x$ -coordinate, and so the distance between Franklin's home and Giizhig's home equals the difference in their  $y$ -coordinates, or 80 m, as shown.

Franklin and Giizhig meet halfway between their two homes, which is a distance of 40 m from each home.

From part (b), it takes Franklin  $\frac{13}{4}$  (or 3.25) minutes to walk straight home from school when walking at a speed of 80 m/min. It takes Franklin an additional  $\frac{40 \text{ m}}{80 \text{ m/min}} = \frac{1}{2}$  (or 0.5) minutes to walk from his home to the halfway point between their two homes.



Thus, it takes Franklin  $\frac{13}{4} + \frac{1}{2} = \frac{15}{4}$  (or 3.75) minutes to walk from the school to his home and then to the halfway point between their two homes.

Since Giizhig leaves the school at the same time as Franklin, and they meet halfway between the two homes, then it also takes Giizhig  $\frac{15}{4}$  minutes to walk from the school to her home and then to the halfway point between their homes.

As in part (a),  $\triangle OAG$  is right-angled at  $A$ , and so by the Pythagorean Theorem,  $GO^2 = OA^2 + AG^2$  or  $GO^2 = (240 \text{ m})^2 + (180 \text{ m})^2 = 90\,000 \text{ m}^2$  and so  $GO = \sqrt{90\,000 \text{ m}^2} = 300 \text{ m}$ .

Therefore, the distance along the straight path from the school to Giizhig's home to the halfway point between the two homes is  $300 \text{ m} + 40 \text{ m} = 340 \text{ m}$ .

Since Giizhig's speed,  $g$  m/min, equals distance divided by time, then

$$g = 340 \div \frac{15}{4} = 340 \times \frac{4}{15} = \frac{272}{3} \text{ or } 90\frac{2}{3}.$$

3. (a) When  $R_3$  is performed on the list 5, 2, 3, 1, 4, 6, the order of the first three numbers in the list is reversed, and so the new list is 3, 2, 5, 1, 4, 6.
- (b) The two Reverse Operations change the order of the first four numbers in the list, 1, 2, 3, 4, while the last two numbers, 5 and 6, do not change. Thus, it is reasonable to consider

that the first Reverse Operation may be  $R_4$ .

After performing  $R_4$  on the original list, the list becomes 4, 3, 2, 1, 5, 6.

Performing  $R_2$  on this list gives the final list 3, 4, 2, 1, 5, 6.

Thus, the two Reverse Operations are  $R_4$  followed by  $R_2$ .

(It is worth noting that there are no other ways to do this using exactly two Reverse Operations.)

- (c) (i) Each Reverse Operation, with the exception of  $R_6$ , does not change the number in the last position of the list. Further,  $R_6$  reverses the order of the entire list and so the first number in the list becomes the last number in the list.

That is, the number 3 moves to the last position only when  $R_6$  is performed on a list in which 3 is in the first position.

Is it possible to perform a Reverse Operation on the original list that moves 3 to the first position of the list?

If  $R_3$  is performed on the original list, 1, 2, 3, 4, 5, 6, the list becomes 3, 2, 1, 4, 5, 6.

If  $R_6$  is then performed on this list, the list becomes 6, 5, 4, 1, 2, 3, as required.

Thus, performing  $R_3$  followed by  $R_6$  results in 3 ending up in the last position of the list and so 2 Reverse Operations achieve the desired result.

- (ii) Since in (i) we found a way to achieve the desired result using 2 Reverse Operations, we need only to explain why performing 1 Reverse Operation is not possible.

As described in (i), the number 3 moves to the last position only when  $R_6$  is performed on a list in which 3 is in the first position. Since 3 is not in the first position of the original list, then performing 1 Reverse Operation can never achieve the desired result.

Together, (i) and (ii) show that the minimum number of Reverse Operations that need to be performed on the list 1, 2, 3, 4, 5, 6 so that 3 ends up in the last position is 2.

- (d) Performing the Reverse Operations  $R_5$ ,  $R_2$ ,  $R_6$ , in order, changes the original list from 1, 2, 3, 4, 5, 6 to 5, 4, 3, 2, 1, 6, and then to 4, 5, 3, 2, 1, 6, and finally to 6, 1, 2, 3, 5, 4.

Thus, 3 Reverse Operations can achieve a list of the desired form.

Can the desired result be achieved by performing fewer than 3 Reverse Operations?

As similarly demonstrated in (c), performing the Reverse Operations  $R_4$  and  $R_6$ , in order on the original list, will move the number 4 to the last position in the list.

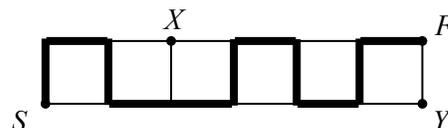
This is the minimum number of Reverse Operations needed to move 4 to the last position in the list and further, these are the only 2 Reverse Operations that move 4 to the last position in the list. (Can you see why?)

However, performing  $R_4$  and  $R_6$ , in order, changes the original list from 1, 2, 3, 4, 5, 6 to 4, 3, 2, 1, 5, 6, and then to 6, 5, 1, 2, 3, 4. The second last number in this list is not 5, and so it is not possible to achieve the desired result in 2 Reverse Operations.

Since it is clearly not possible to achieve the desired result in 1 Reverse Operation, then the minimum number of Reverse Operations is 3.

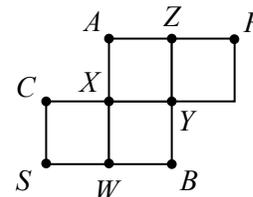
The 3 Reverse Operations  $R_6$ ,  $R_3$ ,  $R_6$  also achieves a list of the desired form. Performing  $R_6$ ,  $R_3$ ,  $R_6$ , in order, changes the original list from 1, 2, 3, 4, 5, 6 to 6, 5, 4, 3, 2, 1 to 4, 5, 6, 3, 2, 1, and finally to 1, 2, 3, 6, 5, 4.

4. (a) The  $SF$  path that passes through each vertex except  $X$  and  $Y$  is shown.



- (b) We begin by labelling the vertices as shown.

Beginning at  $S$ , a path may first proceed right to  $W$  or up to  $C$ . Assume the path first proceeds from  $S$  to  $W$ . In this case, the path may only pass through  $C$  by proceeding from  $X$  to  $C$ .

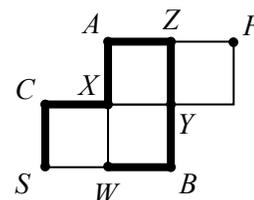


At this point, each edge from  $C$  leads to a vertex that has already been visited. That is, it is not possible for a path that begins by proceeding from  $S$  to  $W$  to pass through  $C$ .

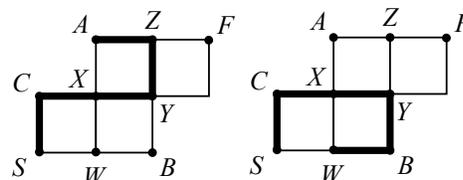
Thus, it must be that if an  $SF$  path is to pass through  $C$ , it does so by first proceeding from  $S$  to  $C$ .

From  $C$ , the path must proceed to  $X$  (since it cannot return to  $S$ ), and so every  $SF$  path that passes through  $C$  must begin  $S - C - X$ . Continuing from this point  $X$ , there are exactly 3 directions in which the path may proceed: up to  $A$ , right to  $Y$ , or down to  $W$ . Next, we consider each of these three possible cases separately.

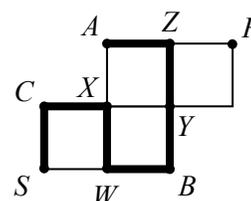
Proceeding up to  $A$  with a goal of passing through  $A$  and  $B$ , the path must proceed  $X - A - Z - Y - B - W$ . Each edge from  $W$  leads to a vertex that has already been visited and so proceeding up from  $X$  to  $A$  is not possible, as shown.



Proceeding from  $X$  to  $Y$  with a goal of passing through  $A$  and  $B$ , the path must proceed  $X - Y - Z - A$  or  $X - Y - B - W$ . Each edge from  $A$  leads to a vertex that has already been visited, as does each edge from  $W$ , and so proceeding right from  $X$  to  $Y$  is not possible, as shown.



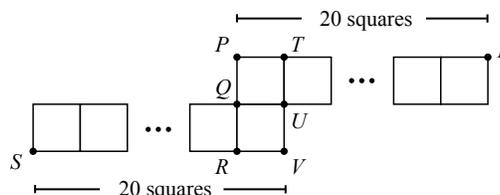
Finally, proceeding down from  $X$  to  $W$  with a goal of passing through  $A$  and  $B$ , the path must proceed  $X - W - B - Y - Z - A$ . Each edge from  $A$  leads to a vertex that has already been visited and so proceeding down from  $X$  to  $W$  is not possible, as shown.



Therefore, there is no  $SF$  path that passes through all three of the vertices  $A$ ,  $B$  and  $C$ .

- (c) We define the “middle column” by labelling the points  $P, Q, R, T, U, V$ , as shown.

Each  $SF$  path must pass through at least one of  $Q$  or  $R$ , and must also pass through at least one of  $T$  or  $U$ .



Each  $SF$  path “enters” the middle column by coming into  $Q$  from the left or by coming into  $R$  from the left.

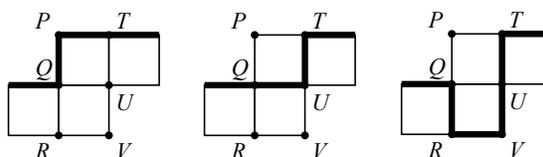
Each  $SF$  path “leaves” the middle column by going out of  $T$  to the right or by going out of  $U$  to the right.

Each  $SF$  path through the middle column falls into exactly one of four groups of paths.

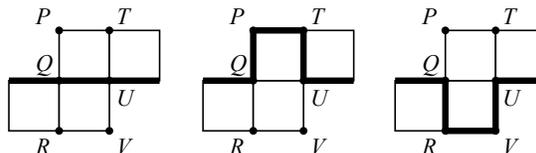
We define these four groups as follows:

- QT path: The portion of an  $SF$  path that comes into  $Q$  from the left and goes out of  $T$  to the right.
- QU path: The portion of an  $SF$  path that comes into  $Q$  from the left and goes out of  $U$  to the right.
- RT path: The portion of an  $SF$  path that comes into  $R$  from the left and goes out of  $T$  to the right.
- RU path: The portion of an  $SF$  path that comes into  $R$  from the left and goes out of  $U$  to the right.

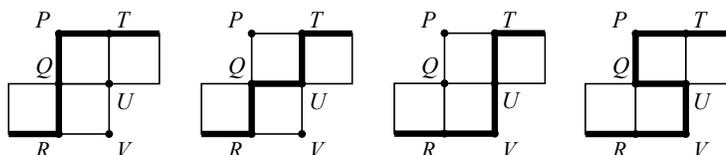
There are exactly 3  $QT$  paths. These are:  $Q - P - T$ ,  $Q - U - T$  and  $Q - R - V - U - T$ .



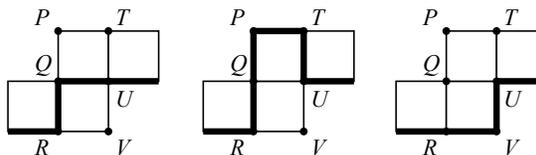
There are exactly 3  $QU$  paths. These are:  $Q - U$ ,  $Q - P - T - U$  and  $Q - R - V - U$ .



There are exactly 4  $RT$  paths. These are:  $R - Q - P - T$ ,  $R - Q - U - T$ ,  $R - V - U - T$ , and  $R - V - U - Q - P - T$ .



There are exactly 3  $RU$  paths. These are:  $R - Q - U$ ,  $R - Q - P - T - U$  and  $R - V - U$ .

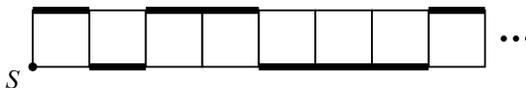


In each of these cases, the left endpoint ( $Q$  or  $R$ ) and the right endpoint ( $T$  or  $U$ ) is attached to a horizontal path segment of the adjacent square.

This leaves exactly 18 squares on each side of the middle column through which the path travels.

In the first 18 squares, an  $SF$  path can proceed horizontally either along the bottom edge of a square or along its top edge.

For example, the diagram below shows a possible selection of top and bottom edges for the first 8 squares.



These choices of top or bottom edges uniquely determine the path since it is forced to proceed along vertical edges exactly when the path through two adjacent squares has different horizontal segments (one follows a top edge and the other a bottom).

The diagram below shows where these vertical edges must be added to the previous example. There is no choice in the selection of vertical edges once the horizontal edges have been chosen.



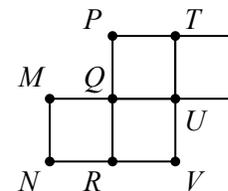
It is worth noting that a path cannot proceed along both the top and bottom edges of a given square since such a path would visit a vertex more than once.

Two choices (top or bottom edge) in each of the first 18 squares gives  $2^{18}$  paths.

Let  $M$  be the point left of  $Q$  and let  $N$  be the point left of  $R$ , as shown.

The  $2^{18}$  paths finish at either  $M$  or  $N$ , and in each case they arrive at the point from the left. That is, the vertical segment  $MN$  is not included in each of the  $2^{18}$  paths.

How many of these  $2^{18}$  paths “connect” to a  $QT$  path (for example)?



Each of the paths arriving at  $N$  (from the left) may proceed to  $M$  and then to  $Q$ , and each of the paths arriving at  $M$  (from the left) may proceed directly to  $Q$ .

Thus, there are exactly  $2^{18}$  paths that enter the middle column through  $Q$  (from the left) or  $2^{18}$  paths begin at  $S$  and arrive at  $Q$  from the left.

Similarly, there are  $2^{18}$  that paths begin at  $S$  and arrive at  $R$  (from the left).

Similar arguments show that there are  $2^{18}$  paths through the last 18 squares, and so there are  $2^{18}$  paths that leave  $T$  to the right, and end at  $F$ , and  $2^{18}$  paths that leave  $U$  to the right, and end at  $F$ .

From our earlier analysis, there are exactly 3  $QT$  paths, and so there are  $2^{18} \cdot 3 \cdot 2^{18} = 3 \cdot 2^{36}$   $SF$  paths that proceed through a  $QT$  path.

There are  $3 \cdot 2^{36}$   $SF$  paths that proceed through a  $QU$  path,  $4 \cdot 2^{36}$  through an  $RT$  path, and  $3 \cdot 2^{36}$  through an  $RU$  path.

Thus, there is a total of  $2^{36}(3 + 3 + 4 + 3)$  or  $13 \cdot 2^{36}$   $SF$  paths.



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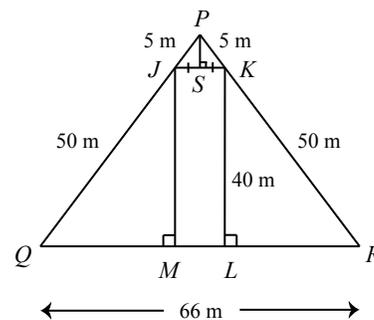
## ***2020 Fryer Contest***

**Wednesday, April 15, 2020**  
(in North America and South America)

**Thursday, April 16, 2020**  
(outside of North America and South America)

*Solutions*

1. (a) Locating point  $C$  on the graph, Cao pays \$7.00 for 14 posters which is a price of  $\$7 \div 14 = \$0.50$  per poster.
- (b) From part (a), Daniel pays \$1.60 per poster and Cao pays \$0.50 per poster. Calculating, Annie pays \$10.00 for 5 posters or  $\$10 \div 5 = \$2.00$  per poster, Bogdan pays  $\$8.00 \div 8 = \$1.00$  per poster, and Emily pays  $\$15.00 \div 15 = \$1.00$  per poster. Thus, Bogdan and Emily are paying the same price per poster. (Alternatively, we may have noticed that the line from the origin  $(0, 0)$  passing through  $E$  also passes through  $B$ . The slope of this line represents the price in dollars per poster for Bogdan and for Emily.)
- (c) From part (a), Daniel paid the company that printed his first batch \$1.60 per poster. To print his second batch at the local library, Daniel would pay  $\$60.00 \div 40 = \$1.50$  per poster. To spend less money, Daniel should print his second batch at the library. Alternatively, Daniel paid \$16.00 for his first batch of 10 posters and thus would pay  $4 \times \$16.00 = \$64.00$  to print his second batch of 40 posters using the same company that printed his first batch. Since the cost to print the 40 posters at the library is \$60.00, he should print them at the library to save money.
- (d) In part (b), we calculated that Emily's printing company charges her \$1.00 per poster. Since this is a fixed price per poster, then Emily paid \$1.00 per poster for each of the 25 posters that she had printed, for a total of \$25.00 spent. Annie and Emily each printed 25 posters and spent the same amount of money, \$25.00. Annie paid \$10.00 to print her first 5 posters, and so she spent  $\$25.00 - \$10.00 = \$15.00$  to have her additional  $25 - 5 = 20$  posters printed. Thus, Annie was charged  $\$15.00 \div 20 = \$0.75$  per additional poster.
2. (a) In  $\triangle KLR$ , we have  $\angle KLR = 90^\circ$  and by using the Pythagorean Theorem, we get  $LR^2 = 50^2 - 40^2 = 900$  and so  $LR = \sqrt{900} = 30$  m (since  $LR > 0$ ).
- (b) We begin by showing that  $\triangle JMQ$  is congruent to  $\triangle KLR$ . Since  $JKLM$  is a rectangle, then  $JM = KL = 40$  m. In addition, hypotenuse  $JQ$  has the same length as hypotenuse  $KR$ , and so  $\triangle JMQ$  is congruent to  $\triangle KLR$  by HS congruence. Thus,  $MQ = LR = 30$  m and so  $ML = 66 - 30 - 30 = 6$  m.
- (c) Since  $PJ = PK = 5$  m,  $\triangle PJK$  is isosceles and so the height,  $PS$ , drawn from  $P$  to  $JK$  bisects  $JK$ , as shown. Since  $JKLM$  is a rectangle, then  $JK = ML = 6$  m and so  $SK = \frac{JK}{2} = 3$  m. Using the Pythagorean Theorem in  $\triangle PSK$ , we get  $PS^2 = 5^2 - 3^2 = 16$  and so  $PS = 4$  m (since  $PS > 0$ ). Thus the height of  $\triangle PJK$  drawn from  $P$  to  $JK$  is 4 m.



- (d) We begin by determining the area of
- $\triangle PQR$
- .

Construct the height of  $\triangle PQR$  drawn from  $P$  to  $T$  on  $QR$ , as shown.

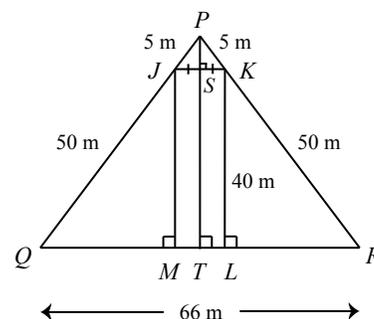
Since  $PT$  is perpendicular to  $QR$ , then  $PT$  is parallel to  $KL$  (since  $KL$  is also perpendicular to  $QR$ ).

By symmetry,  $PT$  passes through  $S$ , and so the height  $PT$  is equal to  $PS + ST = PS + KL$  or  $4 + 40 = 44$  m.

The area of  $\triangle PQR$  is  $\frac{1}{2} \times QR \times PT = \frac{1}{2} \times 66 \times 44 = 1452$  m<sup>2</sup>.

The area of  $JKLM$  is  $ML \times KL = 6 \times 40 = 240$  m<sup>2</sup>.

The fraction of the area of  $\triangle PQR$  that is covered by rectangle  $JKLM$  is  $\frac{240}{1452} = \frac{20}{121}$ .



3. (a) If the 5
- <sup>th</sup>
- term in a Dlin sequence is 142, then the 6
- <sup>th</sup>
- term is
- $(142 + 1) \times 2 = 143 \times 2 = 286$
- .

To determine the 4<sup>th</sup> term in the sequence given the 5<sup>th</sup>, we “undo” adding 1 followed by doubling the result by first dividing the 5<sup>th</sup> term by 2 and then subtracting 1 from the result.

To see this, consider that if two consecutive terms in a Dlin sequence are  $a$  followed by  $b$ , then  $b = (a + 1) \times 2$ .

To determine the operations needed to find  $a$  given  $b$  (that is, to move backward in the sequence), we rearrange this equation to solve for  $a$ .

$$\begin{aligned} b &= (a + 1) \times 2 \\ \frac{b}{2} &= a + 1 \\ \frac{b}{2} - 1 &= a \end{aligned}$$

Thus if the 5<sup>th</sup> term in the sequence is 142, then the 4<sup>th</sup> term is  $\frac{142}{2} - 1 = 71 - 1 = 70$ . (We may check that the term following 70 is indeed  $(70 + 1) \times 2 = 142$ .)

- (b) If the 1
- <sup>st</sup>
- term is 1406, then clearly this is a Dlin sequence that includes 1406.

If the 2<sup>nd</sup> term in a Dlin sequence is 1406, then the 1<sup>st</sup> term in the sequence is  $\frac{1406}{2} - 1 = 703 - 1 = 702$ .

If the 3<sup>rd</sup> term in a Dlin sequence is 1406, then the 2<sup>nd</sup> term is 702 (as calculated in the line above) and the 1<sup>st</sup> term in the sequence is  $\frac{702}{2} - 1 = 351 - 1 = 350$ .

If the 4<sup>th</sup> term in a Dlin sequence is 1406, then the 3<sup>rd</sup> term is 702, the 2<sup>nd</sup> term is 350, and the 1<sup>st</sup> term in the sequence is  $\frac{350}{2} - 1 = 175 - 1 = 174$ .

At this point, we see that 174, 350, 702, and 1406 are possible 1<sup>st</sup> terms which give a Dlin sequence that includes 1406.

We may continue this process of working backward (dividing by 2 and subtracting 1) to determine all possible 1<sup>st</sup> terms which give a Dlin sequence that includes 1406.

$$1406 \rightarrow 702 \rightarrow 350 \rightarrow 174 \rightarrow \frac{174}{2} - 1 = 86 \rightarrow \frac{86}{2} - 1 = 42 \rightarrow \frac{42}{2} - 1 = 20 \rightarrow \frac{20}{2} - 1 = 9$$

Attempting to continue the process beyond 9 gives  $\frac{9}{2} - 1 = \frac{7}{2}$  which is not possible since the 1<sup>st</sup> term in a Dlin sequence must be a positive integer (and so all terms are positive integers).

Thus, the possible 1<sup>st</sup> terms which give a Dlin sequence that includes 1406 are 9, 20, 42, 86, 174, 350, 702, and 1406.

- (c) Each of the integers from 10 to 19 inclusive is a possible first term, and so we must determine the ones digit of each term which follows each of these ten possible first terms. If the 1
- <sup>st</sup>
- term is 10, then the 2
- <sup>nd</sup>
- term
- $(10 + 1) \times 2 = 22$
- has ones digit 2, and the 3
- <sup>rd</sup>
- term

$(22 + 1) \times 2 = 46$  has ones digit 6.

If the 1<sup>st</sup> term is 11, then the ones digit of the 2<sup>nd</sup> term  $(11 + 1) \times 2 = 24$  is 4, and the 3<sup>rd</sup> term  $(24 + 1) \times 2 = 50$  has ones digit 0.

Given each of the possible first terms, we list the ones digits of the 2<sup>nd</sup> and 3<sup>rd</sup> terms in the table below.

1 <sup>st</sup> term	10	11	12	13	14	15	16	17	18	19
Units digit of the 2 <sup>nd</sup> term	2	4	6	8	0	2	4	6	8	0
Units digit of the 3 <sup>rd</sup> term	6	0	4	8	2	6	0	4	8	2

From the table above, we see that the only ones digit which repeats itself is 8.

Thus, if the 1<sup>st</sup> term in the sequence is 18 (has ones digit 8), then the 2<sup>nd</sup> and 3<sup>rd</sup> terms in the sequence have ones digit 8 and so all terms will have the same ones digit, 8.

Similarly, if the 1<sup>st</sup> term in the sequence is 13 (has ones digit 3), then the 2<sup>nd</sup> and 3<sup>rd</sup> terms in the sequence have ones digit 8.

It then follows that all further terms after the first will have ones digit 8.

The 1<sup>st</sup> terms (from 10 to 19 inclusive) which produce a Dlin sequence in which all terms after the 1<sup>st</sup> term have the same ones digit are 13 and 18.

- (d) If the 1<sup>st</sup> term in a Dlin sequence is  $x$ , then the 2<sup>nd</sup> term is  $(x + 1) \times 2 = 2x + 2$ , and the 3<sup>rd</sup> term is  $(2x + 2 + 1) \times 2 = (2x + 3) \times 2 = 4x + 6$ .

For example, if  $x = 1$  (note that this is the smallest possible 1<sup>st</sup> term of a Dlin sequence), then the 3<sup>rd</sup> term is  $4 \times 1 + 6 = 10$ , and if  $x = 2$ , the 3<sup>rd</sup> term is  $4 \times 2 + 6 = 14$ .

What is the largest possible value of  $x$  (the 1<sup>st</sup> term of the sequence) which makes  $4x + 6$  (the 3<sup>rd</sup> term of the sequence) less than or equal to 2020?

Setting  $4x + 6$  equal to 2020 and solving, we get  $4x = 2014$  and so  $x = 503.5$ .

Since the 1<sup>st</sup> term of the sequence must be a positive integer, the 3<sup>rd</sup> term cannot be 2020. Similarly, solving  $4x + 6 = 2019$ , we get that  $x$  is not an integer and so the 3<sup>rd</sup> term of a Dlin sequence cannot equal 2019.

When  $4x + 6 = 2018$ , we get  $4x = 2012$  and so  $x = 503$ .

Thus, if a Dlin sequence has 1<sup>st</sup> term equal to 503, then the 3<sup>rd</sup> term of the sequence is a positive integer between 1 and 2020, namely 2018.

Further, 503 is the largest possible 1<sup>st</sup> term for which the 3<sup>rd</sup> term has this property.

Each 1<sup>st</sup> term  $x$  will give a different 3<sup>rd</sup> term,  $4x + 6$ .

Thus, to count the number of positive integers between 1 and 2020, inclusive, that can be the 3<sup>rd</sup> term in a Dlin sequence, we may count the number of 1<sup>st</sup> terms which give a 3<sup>rd</sup> term having this property.

The smallest possible 1<sup>st</sup> term is 1 (giving a 3<sup>rd</sup> term of 10) and the largest possible 1<sup>st</sup> term is 503 (which gives a 3<sup>rd</sup> term of 2018).

Further, every value of  $x$  between 1 and 503 gives a different 3<sup>rd</sup> term between 10 and 2018.

Thus, there are 503 positive integers between 1 and 2020, inclusive, that can be the 3<sup>rd</sup> term in a Dlin sequence.

4. (a) There are 10 different ways to colour a  $5 \times 1$  grid so that exactly 3 cells are red and 2 cells are blue.

One way to see this is to count the number of different ways to colour 2 cells blue, since each of the 3 remaining cells must be coloured red.

(Alternatively, we could count the number of different ways to colour 3 cells red.)

There are 5 cells that may be chosen first to be coloured blue.

After this chosen cell is coloured, there are 4 remaining cells that can be chosen to be

coloured blue. Thus, there are  $5 \times 4 = 20$  such choices of cells to be coloured blue. However, since the two blue cells are indistinguishable from one another, we have counted each of the different ways of colouring 2 cells blue twice.

For example, choosing to first colour the cell in the second row blue and then colouring the cell in the fifth row blue gives the same colouring as first choosing to colour the cell in the fifth row blue and then colouring the cell in the second row blue.

Thus, there are  $20 \div 2 = 10$  different ways that a  $5 \times 1$  grid can be coloured so that exactly 3 cells are red and 2 cells are blue. These 10 colourings are shown to the right.

B	B	B	B	R
B	R	R	R	B
R	B	R	R	B
R	R	B	R	R
R	R	R	B	R
R	R	R	R	R
B	B	R	R	R
R	R	B	B	R
B	R	B	R	B
R	B	R	B	B

- (b) We begin by observing that there are exactly 2 different ways to colour each of the 13 cells (red or blue), and so there are a total of  $2^{13}$  possible colourings of a  $1 \times 13$  grid.

For each  $1 \times 13$  grid, Carrie counts the number of cells coloured red, call this number  $r$ , and she counts the number of cells coloured blue, call this number  $b$ .

Since there are 13 cells in total, then  $r + b = 13$  and either  $r > b$  or  $b > r$  ( $r$  and  $b$  cannot be equal since they are integers and their sum is odd).

If  $r > b$ , then  $r > 13 - r$  which gives  $2r > 13$  or  $r > 6.5$  and so  $r \geq 7$  (since  $r$  is an integer).

If  $b > r$ , then  $b > 13 - b$  which gives  $2b > 13$  or  $b > 6.5$  and so  $b \geq 7$  (since  $b$  is an integer). Thus, if the number of cells coloured red is greater than the number of cells coloured blue, then Carrie writes down the number of red cells (since she writes the maximum of  $r$  and  $b$ ), and this number is at least 7.

Alternatively, if the number of cells coloured blue is greater than the number of cells coloured red, then Carrie writes down the number of blue cells, and this number is at least 7.

In either case, each number that Carrie writes down in her list is between 7 and 13, inclusive.

At least one of the possible  $2^{13}$  different colourings of a  $1 \times 13$  grid has 7 cells coloured red and 6 cells coloured blue, and so Carrie's list includes at least one 7.

Since Carrie's list includes a 7 and every number in her list is 7 or greater, then the smallest number in her list is 7.

- (c) In a  $3 \times n$  grid, each column contains exactly 3 cells.

Each of these 3 cells can be coloured in one of two ways, either red or blue.

Since there are two ways that each of the 3 cells may be coloured, each column in a  $3 \times n$  grid can be coloured in one of  $2^3 = 8$  different ways.

Thus, it is possible to colour a  $3 \times n$  grid for integers  $n$ ,  $1 \leq n \leq 8$ , so that each column is coloured in a different way (such an example of a  $3 \times 8$  grid is shown).

R	R	R	B	R	B	B	B
R	R	B	R	B	R	B	B
R	B	R	R	B	B	R	B

Since there are only 8 different ways to colour the 3 cells in any column, then every colouring of a  $3 \times 9$  grid must have at least two columns that are coloured in an identical way. Thus, the smallest possible value of  $n$  is 9.

- (d) The given statement is true.

In a  $5 \times 41$  grid, there are 5 rows and so each of the 41 columns has 5 cells.

In each column, at least 3 of the 5 cells must be the same colour.

In each column, either there are more cells coloured red than blue or there are more cells coloured blue than red (they can't be equal in number since 5 is odd).

In each column, if there are more cells coloured red than blue, we call that column a red column.

Alternatively, if there are more cells coloured blue than red, we call that column a blue column.

Let the total number of red columns be  $\mathbb{R}$  and the number of blue columns be  $\mathbb{B}$ , and so  $\mathbb{R} + \mathbb{B} = 41$ .

Similar to the argument in part (b), if  $\mathbb{R} > \mathbb{B}$ , then  $\mathbb{R} \geq 21$ , otherwise  $\mathbb{B} \geq 21$ .

Assume that  $\mathbb{R} \geq 21$  (the argument that follows can be made in a similar way if  $\mathbb{B} \geq 21$ ).

In this case, each of these 21 (or more) columns has more cells coloured red than those that are coloured blue and so each of these columns has at least 3 cells coloured red.

We will show that the location of 3 cells coloured red is the same in at least 3 of 21 columns.

Of all red columns, consider the first 21 (there are at least 21).

Moving from top to bottom within each of these red columns, consider the first 3 red cells (there are at least 3 cells coloured red).

What is the maximum number of ways to colour each column so that exactly 3 cells are coloured red?

Since a  $5 \times 1$  grid is the size of each column in a  $5 \times 41$  grid, this is the question that was answered in part (a), and so there are 10 such ways.

In the 21 red columns, assume that at most 2 columns have the same 3 cells coloured red. Since there are only 10 different ways to colour each column so that exactly 3 cells are coloured red, then there are at most  $2 \times 10 = 20$  such columns.

However, there are 21 red columns and so we have a contradiction.

Thus, our assumption that in the 21 red columns, there are at most 2 columns in which the same 3 cells are coloured red was incorrect, and so there must be at least 3 columns which have the same 3 cells that are coloured red.

The 9 cells located at the intersection of these 3 columns and the 3 rows containing the red cells within these columns are all coloured red.



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## ***2019 Fryer Contest***

**Wednesday, April 10, 2019**  
(in North America and South America)

**Thursday, April 11, 2019**  
(outside of North America and South America)

*Solutions*

1. (a) The rectangle in Figure A has dimensions  $7 \times 8$ , and thus has perimeter  $2 \times 7 + 2 \times 8$  or 30.

(b) *Solution 1*

We begin by labelling Figure B as shown.

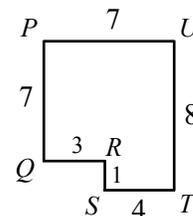
The width of Figure B is  $PU = 7$ .

However, the width is also equal to  $QR + ST$ .

Since  $QR = 3$ , then  $ST = 7 - 3 = 4$ .

Similarly,  $PQ + RS = UT = 8$  and since  $RS = 1$ , then  $PQ = 7$ .

The perimeter of Figure B is  $PQ + QR + RS + ST + TU + UP = 7 + 3 + 1 + 4 + 8 + 7 = 30$ .



*Solution 2*

The answer in Solution 1, 30, is equal to the answer in part (a). Why?

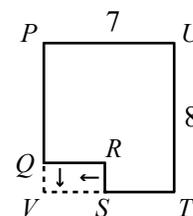
Consider sliding  $RS$  horizontally left to  $QV$ , and sliding  $QR$  vertically down to  $VS$ , as shown.

Since  $QRSV$  is a rectangle, then  $PVTU$  is a rectangle.

Further, since  $RS = QV$  and  $QR = VS$ , then the perimeter of Figure B is equal to the perimeter of rectangle  $PVTU$ .

Since  $PV = UT = 8$  and  $VT = PU = 7$ , the perimeter of rectangle  $PVTU$  is 30 (this is the rectangle from part (a)).

Therefore, the perimeter of Figure B is 30.



- (c) Following the approach in Solution 2 above, the perimeter of Figure C is equal to the perimeter of a rectangle with side lengths  $k + 4$  and  $k + 2$ .

Since the perimeter of Figure C is 56, then  $2(k + 4) + 2(k + 2) = 56$  or  $2k + 8 + 2k + 4 = 56$  and so  $4k = 44$  or  $k = 11$ .

(Alternatively, we could have determined that the two missing lengths in Figure C are each equal to  $k$ , and then found the perimeter of Figure C,  $4k + 12$ , by adding the lengths of the six segments.)

- (d) Each of the segments having lengths 4 or 7 may be “pushed out” to show that the perimeter of Figure D is equal to the perimeter of a square with side length  $8n + 1$ .

That is, the perimeter of Figure D is  $4(8n + 1) = 32n + 4$ .

We require the largest integer  $n$  for which  $32n + 4 < 1000$ .

Solving this inequality, we get  $32n < 996$  or  $n < \frac{996}{32}$  and so  $n < 31.125$ .

Thus, the largest integer  $n$  for which the perimeter of Figure D is less than 1000 is 31.

2. (a) In 10 minutes, there are  $10 \times 60 = 600$  seconds.

If the machine is set to make one cut every 8 seconds, then  $\frac{600 \text{ s}}{8 \text{ s}} = 75$  pieces of rope will be cut off in 10 minutes.

- (b) Rope is fed into the machine at a constant rate of 2 metres per second.

If the machine is set to make one cut every 3 seconds, then the length of each piece of rope that is cut off is  $2 \times 3 = 6$  metres.

(c) *Solution 1*

Rope is fed into the machine at a constant rate of 2 metres per second, which is equivalent to  $2 \times 60 = 120$  metres per minute.

If each piece of rope that is cut off is 30 m long, then the machine is set to make  $\frac{120 \text{ m}}{30 \text{ m}} = 4$  cuts per minute.

*Solution 2*

Rope is fed into the machine at a constant rate of 2 metres per second.

To cut off a piece of rope 30 m long, the machine must be set to make a cut every  $\frac{30}{2} = 15$  seconds.

If the machine makes a cut every 15 seconds, then it makes  $\frac{60 \text{ s}}{15 \text{ s}} = 4$  cuts per minute.

*Solution 3*

In part (b), the machine was set to make one cut every 3 seconds, and so the length of each piece of rope that is cut off is 6 metres.

Since rope is fed into the machine at a constant rate, to cut off a piece of rope that is 5 times longer (30 m is 5 times longer than 6 m), the machine must be set to make each cut 5 times more slowly, or every  $5 \times 3 = 15$  seconds.

If the machine makes a cut every 15 seconds, then it makes  $\frac{60 \text{ s}}{15 \text{ s}} = 4$  cuts per minute.

(d) *Solution 1*

If the machine is set to make 16 cuts per minute, or 16 cuts every 60 seconds, it will make one cut every  $\frac{60 \text{ s}}{16} = 3.75$  seconds. Rope is fed into the machine at a constant rate of 2 metres per second.

Since the machine is set to make one cut every 3.75 seconds, then the length of each piece of rope that is cut off is  $2 \times 3.75 = 7.5$  metres.

*Solution 2*

If the machine is set to make 16 cuts per minute, then every 60 seconds it is set to make 16 cuts.

The rope is fed into the machine at a constant rate of 2 metres per second, and so after 60 seconds,  $60 \times 2 = 120$  metres of rope have passed through the machine.

During these 60 seconds, the 120 metres of rope is cut 16 times, and so the length of each piece of rope that is cut off is  $\frac{120 \text{ m}}{16} = 7.5$  metres.

3. (a) *Solution 1*

In the list of integers beginning at 1, the 6<sup>th</sup> multiple of 5 is  $6 \times 5 = 30$ .

Thus, Tania has listed each of the integers from 1 to 29 with the exception of the positive multiples of 5 less than 30: 5, 10, 15, 20, 25.

Therefore, just before Tania leaves out the 6<sup>th</sup> multiple of 5, she has listed  $29 - 5 = 24$  integers.

*Solution 2*

Beginning at 1, each group of five integers has one integer that is a multiple of 5.

For example, the first group of five integers, 1, 2, 3, 4, 5 has one multiple of 5 (namely 5), and the second group of five integers, 6, 7, 8, 9, 10 has one multiple of 5 (namely 10).

In Tania's list of integers, she leaves out the integers that are multiples of 5, and so in every group of five integers, Tania lists four of these integers.

Thus, just before Tania leaves out the 6<sup>th</sup> multiple of 5, she has listed  $6 \times 4 = 24$  integers.

(b) *Solution 1*

Tania writes 2019 just before leaving out 2020 (since 2020 is a multiple of 5).

Beginning at 1, 2020 is the 404<sup>th</sup> multiple of 5 since  $\frac{2020}{5} = 404$ .

That is, the integers from 1 to 2020 contain 404 groups of 5 integers.

Each of these 404 groups contain one integer that is a multiple of 5, and so Tania leaves

out 404 integers (including 2020) in the list of all integers from 1 to 2020.

If the  $k^{\text{th}}$  integer in Tania's list is 2019, then  $k = 2020 - 404 = 1616$ .

*Solution 2*

Tania writes 2019 just before leaving out 2020 (since 2020 is a multiple of 5).

Beginning at 1, 2020 is the  $404^{\text{th}}$  multiple of 5 since  $\frac{2020}{5} = 404$ .

That is, the integers from 1 to 2020 contain 404 groups of 5 integers.

In Tania's list of integers, she leaves out the integers that are multiples of 5, and so in every group of five integers, Tania lists four of these integers.

If the  $k^{\text{th}}$  integer in Tania's list is 2019, then  $k = 404 \times 4 = 1616$ .

(c) *Solution 1*

We begin by determining which integers are in Tania's list.

In each successive group of 5 consecutive integers beginning at 1, Tania lists 4 of the integers (since she leaves out each integer that is a multiple of 5).

That is, in each of these groups of 5 integers, Tania's list contains  $\frac{4}{5}$  of the integers.

Consider all positive integers from 1 to  $n$ , where  $n$  is a multiple of 5.

Of these  $n$  integers, Tania's list contains  $\frac{4}{5}n$  integers.

Tania's list contains 200 integers, and so  $\frac{4}{5}n = 200$  or  $n = \frac{200 \times 5}{4} = 250$ .

That is, if Tania lists the positive integers from 1 to 250 and leaves out the integers that are multiples of 5, her list will contain  $\frac{4}{5} \times 250 = 200$  integers.

We are required to determine the sum,  $1 + 2 + 3 + 4 + 6 + \cdots + 244 + 246 + 247 + 248 + 249$ , of the first 250 positive integers with the integers that are multiples of 5 removed.

We will proceed to determine this sum by first calculating the sum of all integers from 1 to 250, and then subtracting from that sum all integers in this list that are multiples of 5.

The sum of the integers from 1 to  $n$  is given by  $\frac{1}{2}n(n+1)$ , and so the sum of the integers from 1 to 250 is equal to  $\frac{1}{2}(250)(251) = 31\,375$ .

The multiples of 5 in this list,  $5 + 10 + 15 + \cdots + 240 + 245 + 250$ , can be written as  $5(1 + 2 + 3 + \cdots + 48 + 49 + 50)$  by removing the common factor 5 (since each is a multiple of 5).

This sum is equal to  $5 \times \frac{1}{2}(50)(51) = 6375$ .

If Tania lists the positive integers, in order, leaving out the integers that are multiples of 5, the sum of the first 200 integers in her list is  $31\,375 - 6375 = 25\,000$ .

*Solution 2*

As was shown in Solution 1, the sum of the first 200 integers in Tania's list is the sum  $1 + 2 + 3 + 4 + 6 + \cdots + 244 + 246 + 247 + 248 + 249$ .

The sum of the first and last integers in this list is  $1 + 249 = 250$ .

The sum of the second integer and the second last integer is  $2 + 248 = 250$ .

The sum of the third integer and the third last integer is  $3 + 247 = 250$ .

We continue in this way moving toward the middle of the list.

That is, we move one number to the right of the previous first number, and one number to the left of the previous second number.

Doing so, we recognize that

- when the first number in the new pair is one more than the previous first number, then the number it is paired with is one less than the previous second number, and
- when the first number in the new pair is two more than the previous first number (as is the case when a multiple of 5 is omitted), then the number it is paired with is two less than the previous second number.

That is, as we continue moving toward the middle of Tania's list, each pair will continue to have a sum equal to 250.

Since there are 200 numbers in Tania's list, there are 100 such pairs, each having a sum equal to 250.

Thus, if Tania lists the positive integers, in order, leaving out the integers that are multiples of 5, the sum of the first 200 integers in her list is  $250 \times 100 = 25\,000$ .

4. (a) We begin by observing that when  $x = 18$ ,

$$12 \times 18 \times 24 = 12 \times (6 \times 3) \times (2 \times 12) = 12 \times 6 \times 6 \times 12 = (12 \times 6)^2 = 72^2.$$

Therefore, 12, 18, 24 is a Shonk sequence.

Next, we will show that  $x = 18$  is the only value for which 12,  $x$ , 24 is a Shonk sequence.

In order for 12,  $x$ , 24 to be a Shonk sequence, we must have  $12 < x < 24$ .

This means  $x$  is one of 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, and 23.

When  $x = 13$ , the product  $12 \times 13 \times 24 = 3744$  is not a perfect square.

This can be checked on a calculator, but if we can understand why it is not a perfect square, we will be able to apply the reasoning again.

Suppose there is some integer  $n$  so that  $n^2 = 12 \times 13 \times 24$ .

This means 13 is a factor of  $n^2$ .

However, 13 is prime, so this means 13 must be a factor of  $n$  itself.

This means  $n^2$  has at least two factors of 13, so  $12 \times 13 \times 24$  must have at least two factors of 13.

However, neither 12 nor 24 has a factor of 13, so we conclude that  $12 \times 13 \times 24$  is not a perfect square.

Therefore, 12, 13, 24 is not a Shonk sequence, so  $x \neq 13$ .

The key to the reasoning above is that 13 is a prime number.

Thus, the same reasoning can be applied to conclude that  $x \neq 17$ ,  $x \neq 19$ , and  $x \neq 23$ .

We now suppose  $x = 14$ . In this case, we have that  $12 \times 14 \times 24$  has a factor of 7.

If  $12 \times 14 \times 24 = n^2$  for some integer  $n$ , then 7 is a factor of  $n^2$ , and since 7 is prime, this means  $n$  must have a factor of 7.

Therefore,  $n^2 = 12 \times 14 \times 24$  has at least two factors of 7, but this is not the case as 14 has only one factor of 7, and neither 12 nor 24 has a factor of 7.

Thus,  $12 \times 14 \times 24$  is not a perfect square, so 12, 14, 24 is not a Shonk sequence and  $x \neq 14$ .

Since  $21 = 3 \times 7$ , the same reasoning can be applied to show that  $x \neq 21$ .

That is, for  $12 \times 21 \times 24$  to be a perfect square, it must have at least two factors of 7, but it only has one.

We have now reduced the list of possible  $x$  values to 15, 16, 18, 20, and 22.

Since 15 and 20 both have a factor of 5, the products  $12 \times 15 \times 24$  and  $12 \times 20 \times 24$  each have exactly one factor of the prime number 5, so they cannot be perfect squares.

This means neither of 12, 15, 24 and 12, 20, 24 are Shonk sequences, so  $x \neq 15$  and  $x \neq 20$ .

Since  $22 = 2 \times 11$ , we can also conclude that  $x \neq 22$  since  $12 \times 22 \times 24$  has exactly one factor of the prime number 11, so  $12 \times 22 \times 24$  is not a perfect square, hence, 12, 22, 24 is not a Shonk sequence.

We have now reduced the possibilities to  $x = 16$  and  $x = 18$ .

We will now show that  $12 \times 16 \times 24$  is not a perfect square.

To do this, we write it as a product of prime numbers:

$$12 \times 16 \times 24 = (2 \times 2 \times 3) \times (2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 3)$$

We see that  $12 \times 16 \times 24$  is a product of only the primes 2 and 3.

In particular, there are nine factors of 2 and two factors of 3.

If  $12 \times 16 \times 24 = n^2$  for some integer  $n$ , then  $n^2$  must therefore have exactly nine factors of 2.

However, each factor of  $n$  has the same number of factors of 2, so  $n^2$  must have an even number of factors of 2.

Since  $12 \times 16 \times 24$  has an odd number of factors of 2, we conclude that  $12 \times 16 \times 24 \neq n^2$  for any integer  $n$ , so is not a perfect square.

Therefore, 12, 16, 24 is not a Shonk sequence so  $x \neq 16$ .

We have ruled out all other possibilities of  $x$ , and thus we conclude that 12,  $x$ , 24 is a Shonk sequence when  $x = 18$  only.

- (b) Building on the reasoning from part (a), we will use the observation that in any perfect square, there are an even number of factors of any given prime number.

For example, 100 is a perfect square and  $100 = 2 \times 2 \times 5 \times 5$ , which is a product with two factors of 2 and two factors of 5.

The example given in the question has  $18^2 = 324$ , which equals  $2 \times 2 \times 3 \times 3 \times 3 \times 3$ , which is a product of two factors of 2, and four factors of 3.

Indeed, if  $n$  is an integer with prime factor  $p$ , then there is a “copy” of  $p$  in each factor of  $n$  occurring in  $n^2$ .

On the other hand, if a number factors into a product of primes where each prime occurs an even number of times, then the primes can be grouped to see that the number is a perfect square.

For example, the number with four factors of 2, two factors of 3, and six factors of 5 is a perfect square since

$$2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = (2 \times 2 \times 3 \times 5 \times 5 \times 5)^2$$

In order for 28,  $y$ ,  $z$ , 65 to be a Shonk sequence, we need  $28 \times y \times z \times 65$  to be a perfect square.

By factoring 28 and 65 into primes, we have that

$$2 \times 2 \times 7 \times y \times z \times 5 \times 13$$

must be a perfect square.

The primes 2, 5, 7, and 13 each occur in this factorization.

From the discussion above, there must be an even number of each of these primes if the product is to be a perfect square.

Therefore, there must be another factor of each of the prime numbers 5, 7, and 13 in the product.

Furthermore, these primes must occur as factors of either  $y$  or  $z$ .

This means that one of  $y$  and  $z$  must have a factor of 5, one of  $y$  and  $z$  must have a factor of 7, and one of  $y$  and  $z$  must have a factor of 13.

There are two variables and three primes, so one of  $y$  and  $z$  must have two of these prime

factors.

Since  $5 \times 13 = 65$  and  $7 \times 13 = 91$ , and  $y$  and  $z$  must each be less than 65, then 5 and 13 cannot occur together as factors of  $y$  or  $z$ , as well 7 and 13 cannot occur together as factors of  $y$  or  $z$ .

Therefore, either 5 and 7 are both factors of  $y$  or 5 and 7 are both factors of  $z$ .

This means one of  $y$  or  $z$  is a multiple of  $5 \times 7 = 35$ .

Any positive multiple of 35 other than 35 itself is larger than 65, so we actually get that  $y = 35$  or  $z = 35$ .

The variable which is not equal to 35 must be a multiple of 13 between 28 and 65.

The only such multiples of 13 are 39 and 52 which are both larger than 35.

This forces  $y = 35$ , and now we know that either  $z = 39$  or  $z = 52$ .

If  $z = 39$ , we get

$$28 \times y \times z \times 65 = (2 \times 2 \times 7) \times (5 \times 7) \times (3 \times 13) \times (5 \times 13)$$

but this has only one factor of 3, so it cannot be a perfect square.

On the other hand, if  $z = 52$ , we have

$$\begin{aligned} 28 \times y \times z \times 65 &= (2 \times 2 \times 7) \times (5 \times 7) \times (2 \times 2 \times 13) \times (5 \times 13) \\ &= (2 \times 2 \times 5 \times 7 \times 13)^2 \\ &= 1820^2 \end{aligned}$$

Therefore, the only Shonk sequence of the form  $28, y, z, 65$  has  $y = 35$  and  $z = 52$ .

- (c) The longest Shonk sequence each of whose terms is an integer between 1 and 12 inclusive has length 9.

An example of such a sequence is 1, 2, 3, 4, 5, 6, 8, 9, 10.

Each term after the first is greater than the previous term, so we need to verify that the product of all terms is a perfect square.

By splitting each term into a product of primes, we get

$$\begin{aligned} &1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 8 \times 9 \times 10 \\ &= 2 \times 3 \times (2 \times 2) \times 5 \times (2 \times 3) \times (2 \times 2 \times 2) \times (3 \times 3) \times (2 \times 5) \\ &= (2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5)^2 \\ &= 720^2 \end{aligned}$$

We will now show that no Shonk Sequence, each of whose terms is between 1 and 12 inclusive, can have length greater than 9.

First, note that the only number between 1 and 12 inclusive with a factor of 7 is 7 itself. Therefore, if 7 were included in the sequence, the product of all terms could not possibly be a perfect square.

Similarly, if 11 were included in the sequence, the product could not possibly be a perfect square since no number other than 11 between 1 and 12 inclusive has a factor of 11.

This means the sequence must be made up of the numbers 1, 2, 3, 4, 5, 6, 8, 9, 10, and 12.

There are 10 numbers in this list, so the only way to have a longer Shonk sequence than the one above is for it to include all 10 of these integers.

However, the product

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 8 \times 9 \times 10 \times 12$$

has five factors of 3 (one coming from each of 3, 6, and 12, and two coming from 9).

Therefore, it cannot be a perfect square since a perfect square must have an even number of occurrences of any prime factor.

(d) We need two related facts about perfect squares:

- (F1) A positive integer  $n$  is a perfect square exactly when each prime factor occurs an even number of times. (We saw this in (b).)
- (F2) Suppose that integers  $r$ ,  $s$  and  $t$  satisfy  $r = st$ . If two of  $r, s, t$  are perfect squares, then the third one is a perfect square. This comes from the fact that the prime factors of two of these occur an even number of times each exactly when the prime factors in their product or (integer) quotient also occur an even number of times.

Suppose that  $m, 1176, n, 48\,400$  is an SDSS. Then

- $m < 1176 < n < 48\,400$
- $m \times 1176 \times n$  is a perfect square, and
- $1176 \times n \times 48\,400$  is a perfect square, and
- $m \times 1176 \times n \times 48\,400$  is a perfect square.

Since  $1176 \times n \times 48\,400$  is a perfect square, and  $m \times (1176 \times n \times 48\,400)$  is a perfect square, then  $m$  is perfect square (by F2).

Also, since  $48\,400 = 220^2$  and  $1176 = 6 \times 14^2$ , then  $1176 \times n \times 48\,400 = (220 \times 14)^2 \times (6 \times n)$  and this is a perfect square exactly when  $6n$  is a perfect square.

Therefore, if  $m, 1176, n, 48\,400$  is an SDSS, then  $m$  and  $6n$  are both perfect squares.

Further, if  $m$  and  $6n$  are both perfect squares, then  $m \times 1176 \times n$ ,  $1176 \times n \times 48\,400$ , and  $m \times 1176 \times n \times 48\,400$  are all perfect squares.

We can now answer the question by counting the number of pairs of positive integers  $(m, n)$  satisfying  $1 \leq m < 1176$  and  $1176 < n < 48\,400$ , with  $m$  a perfect square and  $6n$  a perfect square.

Note that  $34^2 = 1156$  which is smaller than 1176, and  $35^2 = 1225$  which is larger than 1176, so the possible values of  $m$  are  $1^2, 2^2, 3^2, \dots, 33^2, 34^2$ .

There are 34 possibilities for the value of  $m$ .

We know that  $6 \times 14^2 = 1176$ , which is not larger than 1176, so the smallest possible value of  $n$  is  $6 \times 15^2$ .

Observe that  $6 \times 89^2 = 47\,526 < 48\,400$ , but  $6 \times 90^2 = 48\,600 > 48\,400$ .

Thus, the possible values of  $n$  are

$$6 \times 15^2, 6 \times 16^2, \dots, 6 \times 89^2$$

There are  $89 - 14 = 75$  such values.

Thus, the number of pairs  $(m, n)$  such that  $m, 1176, n, 48\,400$  an SDSS is  $34 \times 75 = 2550$ .



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## ***2018 Fryer Contest***

**Thursday, April 12, 2018**  
(in North America and South America)

**Friday, April 13, 2018**  
(outside of North America and South America)

*Solutions*

1. (a) On Monday, Shane bought 4 boxes of cherries at a cost of  $\$2.00 \times 4 = \$8.00$ .  
 He also bought 3 boxes of plums at a cost of  $\$3.00 \times 3 = \$9.00$ , and 2 boxes of blueberries at a cost of  $\$4.50 \times 2 = \$9.00$ .  
 In total, Shane paid  $\$8.00 + \$9.00 + \$9.00 = \$26.00$ .
- (b) On Wednesday, Shane bought 2 boxes of plums at at cost of  $\$3.00 \times 2 = \$6.00$ .  
 Since Shane spent  $\$22.00$  in total, he spent  $\$22.00 - \$6.00 = \$16.00$  on boxes of cherries.  
 The price of each box of cherries is  $\$2.00$ , and so Shane bought  $\$16.00 \div \$2.00 = 8$  boxes of cherries.
- (c) *Solution 1*  
 On Saturday, Shane gave the cashier  $\$100.00$  and received  $\$14.50$  in change, and so Shane spent  $\$100.00 - \$14.50 = \$85.50$ .  
 Of the  $\$85.50$ , Shane spent  $\$4.50 \times 3 = \$13.50$  on 3 boxes of blueberries.  
 Therefore, Shane spent  $\$85.50 - \$13.50 = \$72.00$  on boxes of plums and boxes of cherries.  
 If Shane bought  $c$  boxes of cherries, then he bought twice as many boxes or  $2c$  boxes of plums.  
 The cost of  $c$  boxes of cherries is  $\$2.00 \times c = \$2c$ .  
 The cost of  $2c$  boxes of plums is  $\$3.00 \times 2c = \$6c$ .  
 Shane spent a total of  $\$2c + \$6c = \$8c$  on boxes of plums and boxes of cherries, and so  $8c = 72$  or  $c = 9$ .  
 Therefore, Shane bought 9 boxes of cherries.

*Solution 2*

As in Solution 1, we begin by determining that Shane spent  $\$72.00$  on boxes of plums and boxes of cherries.

For every 1 box of cherries that Shane bought, he purchased 2 boxes of plums.

The cost of 1 box of cherries and 2 boxes of plums is  $\$2.00 + 2 \times \$3.00 = \$8.00$  and  $\$72.00 \div \$8.00 = 9$ , so Shane bought 9 boxes of cherries (and  $2 \times 9 = 18$  boxes of plums).

Note: In both solutions, we may check that the cost of 9 boxes of cherries, 18 boxes of plums, and 3 boxes of blueberries is  $9 \times \$2.00 + 18 \times \$3.00 + 3 \times \$4.50 = \$85.50$ , and so the correct change from  $\$100.00$  is  $\$14.50$ , as expected.

2. (a) In the first diagram (Paul's Path),  $\triangle ABM$  is right-angled at  $B$ .  
 By the Pythagorean Theorem,  $MA^2 = AB^2 + BM^2$  or  $MA^2 = 105^2 + 100^2 = 21\,025$  or  $MA = \sqrt{21\,025} = 145$  m (since  $MA > 0$ ).
- (b) In the second diagram (Tyler's Path),  $AD = BC = 200$  m and  $DC = AB = 105$  m (since  $ABCD$  is a rectangle).  
 Thus  $PD = AD - AP = 200 - 140 = 60$  m.  
 Also,  $DQ = DC - QC = 105 - 60 = 45$  m.  
 Since  $\triangle PDQ$  is right-angled at  $D$ , using the Pythagorean Theorem, we get  $PQ^2 = PD^2 + DQ^2$  or  $PQ^2 = 60^2 + 45^2 = 5625$  or  $PQ = \sqrt{5625} = 75$  m (since  $PQ > 0$ ).  
 The total distance that Tyler runs is

$$AP + PQ + QC + CB + BA = 140 + 75 + 60 + 200 + 105 = 580 \text{ m.}$$

- (c) The total distance that Paul runs is

$$AD + DC + CM + MA = 200 + 105 + (200 - 100) + 145 = 550 \text{ m.}$$

Tyler runs at a speed of 145 m/min, and so it takes Tyler  $580 \div 145 = 4$  min to finish his path.

Paul begins at the same time as Tyler and finishes his path 1 minute after Tyler, and so Paul takes  $4 + 1 = 5$  min to finish his path.

In this time, Paul runs a total distance of 550 m and so Paul's speed is  $550 \div 5 = 110$  m/min.

3. (a) We determine the  $x$ -intercept by letting  $y = 0$  in the equation  $y = 2x - 6$  and solving for  $x$ .

Thus,  $0 = 2x - 6$  and so  $2x = 6$  or  $x = 3$ .

The  $x$ -intercept of the line with equation  $y = 2x - 6$  is 3.

We determine the  $y$ -intercept by letting  $x = 0$  in the equation  $y = 2x - 6$  and solving for  $y$ .

Thus,  $y = 2(0) - 6$  and so  $y = -6$ .

The  $y$ -intercept of the line with equation  $y = 2x - 6$  is  $-6$ .

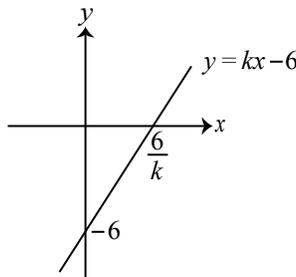
- (b) Letting  $y = 0$ , we get  $0 = kx - 6$  or  $kx = 6$  and so  $x = \frac{6}{k}$ , where  $k \neq 0$ .

The line with equation  $y = kx - 6$  has  $x$ -intercept  $\frac{6}{k}$  ( $k \neq 0$ ).

- (c) From part (b), the line with equation  $y = kx - 6$  has  $x$ -intercept  $\frac{6}{k}$ .

Since  $k > 0$ , then  $\frac{6}{k} > 0$  and so the line intersects the positive  $x$ -axis.

The  $y$ -intercept of the line with equation  $y = kx - 6$  is  $-6$ .



The triangle formed by the line with equation  $y = kx - 6$  ( $k > 0$ ), the positive  $x$ -axis, and the negative  $y$ -axis, has area  $\frac{1}{2} \left( \frac{6}{k} \right) (6) = \frac{36}{2k} = \frac{18}{k}$  (the  $y$ -intercept is  $-6$ , and so the triangle has height 6).

Since the area of this triangle is 6, then  $\frac{18}{k} = 6$  or  $18 = 6k$  and so  $k = 3$ .

- (d) The  $x$ -intercept of the line with equation  $y = 2mx - m^2$  is determined by letting  $y = 0$  and solving for  $x$ .

Thus,  $0 = 2mx - m^2$  or  $0 = m(2x - m)$  and since  $m > 0$ , then  $2x = m$  or  $x = \frac{m}{2}$ .

The  $x$ -intercept of this line is  $\frac{m}{2}$  ( $m > 0$ ).

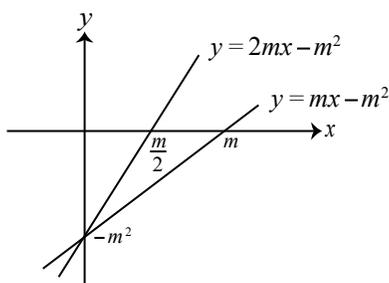
The  $y$ -intercept of the line with equation  $y = 2mx - m^2$  is  $2m(0) - m^2 = -m^2$ .

Similarly, the  $x$ -intercept of the line with equation  $y = mx - m^2$  is given by  $0 = mx - m^2$  or  $0 = m(x - m)$  and since  $m > 0$ , then  $x = m$ .

The  $x$ -intercept of this line is  $m$  ( $m > 0$ ).

The  $y$ -intercept of the line with equation  $y = mx - m^2$  is  $m(0) - m^2 = -m^2$ .

Thus, both lines have the same  $y$ -intercept.



To determine the area of the triangle formed by the positive  $x$ -axis, the line with equation  $y = mx - m^2$ , and the line with equation  $y = 2mx - m^2$  ( $m > 0$ ), we may let the length of the base be the distance between the  $x$ -intercepts or  $m - \frac{m}{2} = \frac{m}{2}$ .

Then the height of this triangle is the perpendicular distance from the  $x$ -axis to the  $y$ -intercept, or  $m^2$  (the  $y$ -intercept is  $-m^2$ , and so the triangle has height  $m^2$ , a positive number).

Therefore, the triangle has area  $\frac{1}{2} \left( \frac{m}{2} \right) (m^2) = \frac{m^3}{4}$ .

The area of this triangle is  $\frac{54}{125}$  and so  $\frac{m^3}{4} = \frac{54}{125}$  or  $m^3 = \frac{216}{125}$  and so  $m = \sqrt[3]{\frac{216}{125}} = \frac{6}{5}$ .

(Note that  $\left(\frac{6}{5}\right)^3 = \frac{216}{125}$ .)

The only value of  $m$  for which the triangle has area  $\frac{54}{125}$  is  $m = \frac{6}{5}$ .

4. (a) There are 2 choices (a 1 or a 2) for each of the 3 digits, and so there are  $2^3 = 8$  3-digit Bauman numbers.

These 3-digit Bauman numbers are: 111, 112, 121, 122, 211, 212, 221, 222.

- (b) Every Bauman number having fewer than three blocks has exactly one block or it has exactly two blocks.

First we consider 10-digit Bauman numbers that have exactly one block.

If a 10-digit Bauman number has exactly 1 block, then the number is made up of 10 ones or it is made up of 10 twos.

Thus, the number of 10-digit Bauman numbers having exactly one block is 2.

Next we consider 10-digit Bauman numbers that have exactly two blocks.

If the Bauman number has exactly two blocks, then it has a block of 1s followed by a block of 2s or it has a block of 2s followed by a block of 1s.

To begin, assume that the block of 1s comes before the block of 2s.

The block of 1s could be 1 digit in length, 2 digits in length, and so on up to 9 digits in length (9 is the maximum length since a block of 2s must follow the block of 1s).

In each case, the remaining digits in the number are all 2s and so there are exactly 9 Bauman numbers of this form.

Similarly, there are 9 Bauman numbers with exactly two blocks whose first block is made up of 2s (from 1 to 9 of them) and whose remaining digits are 1s.

In total, there are  $2 + 9 + 9 = 20$  Bauman numbers consisting of 10-digits and having fewer than three blocks.

- (c) We first consider Bauman numbers that consist of exactly one block.

Any number of 2s will sum to an even number, and so there are no Bauman numbers consisting of a single block of 2s whose digit sum is 7.

There is 1 Bauman number consisting of a single block of seven 1s whose digit sum is 7.

Next, we consider Bauman numbers that consist of exactly two blocks.

In this case, the number must have at least one 2 (since there are two blocks), and at most three 2s, since the digit sum is 7.

If the Bauman number has a block consisting of a single 2, then the second block must consist of five 1s.

There are exactly 2 numbers having this form: 211 111 and 111 112.

If the Bauman number has a block consisting of two 2s, then the second block must consist of three 1s.

There are exactly 2 numbers having this form: 22 111 and 11 122.

If the Bauman number has a block consisting of three 2s, then the second block must consist of one 1.

There are exactly 2 numbers having this form: 2221 and 1222.

There are 6 Bauman numbers consisting of exactly two blocks and whose digit sum is 7.

Finally, we consider Bauman numbers that consist of exactly three blocks.

As before, the number must have at least one 2 (since there are three blocks) and at most three 2s, since the digit sum is 7.

Bauman numbers of this form must consist of:

- (i) one block of one 2 and two blocks of 1s (five 1s in total), or
- (ii) one block of two 2s and two blocks of 1s (three 1s in total), or
- (iii) one block of one 1 and two blocks of 2s (three 2s in total), or
- (iv) one block of three 1s and two blocks of 2s (two 2s in total)

We note that it is not possible for Bauman numbers of this form to consist of:

- one block of three or more 2s and two blocks of 1s since the digit sum would be greater than 7;
- one block of two 1s and two blocks of 2s since the digit sum would be even;
- one block of four or more 1s and two blocks of 2s since the digit sum would be greater than 7.

In the table below, we list the Bauman numbers having exactly 3 blocks and whose digit sum is 7.

Each row of the table corresponds to one of the four possible cases outlined above.

Case	Bauman numbers
(i)	121 111, 112 111, 111 211, 111 121
(ii)	12 211, 11 221
(iii)	2122, 2212
(iv)	21 112

There are 9 Bauman numbers consisting of exactly three blocks and whose digit sum is 7.

The number of Bauman numbers that consist of at most three blocks and have the property that the sum of the digits is 7 is  $1 + 6 + 9 = 16$ .

- (d) We introduce the notation  $\boxed{2}$  to represent a block of exactly 2018 2s, and  $X_n$  to represent a string of  $n$  digits, in which each digit can be either a 1 or a 2.

We are asked to determine the number of 4037-digit Bauman numbers that include at least one  $\boxed{2}$ .

We consider the following three cases.

- (i) The Bauman number begins with a  $\boxed{2}$ . That is, the first 2018 digits are 2s and the 2019<sup>th</sup> digit is a 1. We note that in this case the 2019<sup>th</sup> digit must be a 1, otherwise the block at the beginning of this number would have at least 2019 2s and thus would not begin with a  $\boxed{2}$ .
- (ii) The Bauman number ends with a  $\boxed{2}$ . That is, the last 2018 digits are 2s and the digit preceding these digits (which is again the 2019<sup>th</sup> digit in the number) is a 1. It is worth noting here that both (i) and (ii) can occur simultaneously.
- (iii) The Bauman number contains a  $\boxed{2}$ , however the  $\boxed{2}$  does not occur at the beginning of the number and it does not occur at the end of the number. In this case, the Bauman number contains the 2020 digits  $1\boxed{2}1$ , in this order.

We note that every Bauman number that contains at least one  $\boxed{2}$  must satisfy the conditions in at least one of these three cases above.

Next, we count the number of 4037-digit Bauman numbers that are in each of these three cases.

#### Case (i)

The first 2019 digits of the number are  $\boxed{2}1$ , and so there are  $4037 - 2019 = 2018$  digits remaining, each of which can be either a 1 or a 2.

In this case, the Bauman numbers are of the form  $\boxed{2}1X_{2018}$ .

There are two choices for each of the remaining 2018 digits (each can be a 1 or a 2), and so there are  $2^{2018}$  Bauman numbers of this form.

#### Case (ii)

Similarly, the last 2019 digits of the number are  $1\boxed{2}$ , and so there are  $4037 - 2019 = 2018$  digits remaining, each of which can be a 1 or a 2.

In this case, the Bauman numbers are of the form  $X_{2018}1\boxed{2}$ .

There are two choices for each of the remaining 2018 digits (each can be a 1 or a 2), and so there are  $2^{2018}$  Bauman numbers of this form.

As was noted earlier, there is exactly 1 number which satisfies the conditions of both Case (i) and Case (ii).

The Bauman number  $\boxed{2}1\boxed{2}$  has 4037 digits, and begins and ends with a  $\boxed{2}$ .

That is, we have counted this number twice, once in Case (i) and again in Case (ii).

Therefore, the total number of 4037-digit Bauman numbers that satisfy the conditions of Case (i) or Case (ii) is  $2^{2018} + 2^{2018} - 1$ .

#### Case (iii)

We first note that every Bauman number satisfying the conditions of Case (iii) must be different than every Bauman number satisfying the conditions of Case (i) or Case (ii).

Begin by assuming that the first 2020 digits of the Bauman number are  $1\boxed{2}1$ .

In this case, there are  $4037 - 2020 = 2017$  digits remaining, each of which can be a 1 or a 2.

These Bauman numbers are of the form  $1\boxed{2}1X_{2017}$ .

Since only 2017 digits remain to be chosen, it is not possible to have a second  $\boxed{2}$  since each  $\boxed{2}$  contains 2018 digits.

There are two choices for each of the remaining 2017 digits, and so there are  $2^{2017}$  Bauman numbers of this form.

We may move the string of 2020 digits  $1\boxed{2}1$  to the right 1 digit to give a number having the new form  $X_11\boxed{2}1X_{2016}$ .

Again, there are  $2^{2017}$  ways to replace the 2017 digits.

Continuing to move the string of 2020 digits 1 digit to the right, we get numbers of the form  $X_2 1 \boxed{2} 1 X_{2015}$ ,  $X_3 1 \boxed{2} 1 X_{2014}$ ,  $X_4 1 \boxed{2} 1 X_{2013}$ , and so on until the string  $1 \boxed{2} 1$  is at the end of the number and we obtain a number of the form  $X_{2017} 1 \boxed{2} 1$ .

In each of these, there are two choices for each of the remaining 2017 digits, and so there are  $2^{2017}$  Bauman numbers of this form.

That is, for each Bauman number having one of the forms:

$$1 \boxed{2} 1 X_{2017}, X_1 1 \boxed{2} 1 X_{2016}, X_2 1 \boxed{2} 1 X_{2015}, X_3 1 \boxed{2} 1 X_{2014}, \dots, X_{2016} 1 \boxed{2} 1 X_1, X_{2017} 1 \boxed{2} 1,$$

there are  $2^{2017}$  ways to fill in the remaining digits.

Since there are 2018 of these forms, each having  $2^{2017}$  ways to fill in the remaining digits, then there are  $2018 \cdot 2^{2017}$  Bauman numbers satisfying the conditions of Case (iii).

The number of 4037-digit Bauman numbers that include at least one block of exactly 2018 2s is

$$\begin{aligned} 2^{2018} + 2^{2018} - 1 + 2018 \cdot 2^{2017} &= 2 \cdot 2^{2018} - 1 + 1009 \cdot 2 \cdot 2^{2017} \\ &= 2 \cdot 2^{2018} - 1 + 1009 \cdot 2^{2018} \\ &= 1011 \cdot 2^{2018} - 1. \end{aligned}$$



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## ***2017 Fryer Contest***

**Wednesday, April 12, 2017**  
(in North America and South America)

**Thursday, April 13, 2017**  
(outside of North America and South America)

*Solutions*

1. (a) Red pens are sold in packages of 6 pens.  
 Therefore, 5 packages of red pens contain  $5 \times 6 = 30$  pens.  
 Blue pens are sold in packages of 9 pens.  
 Therefore, 3 packages of blue pens contain  $3 \times 9 = 27$  pens.  
 Altogether, Igor bought  $30 + 27 = 57$  pens.
- (b) Robin bought 21 packages of red pens which contain  $21 \times 6 = 126$  pens.  
 Of the 369 pens that Robin bought, the number of blue pens was  $369 - 126 = 243$ .  
 Blue pens are sold in packages of 9, so the number of packages of blue pens bought by Robin was  $243 \div 9 = 27$ .

(c) *Solution 1*

Let the number of packages of red pens bought by Susan be  $r$ , for some whole number  $r$ .  
 Let the number of packages of blue pens bought by Susan be  $b$ , for some whole number  $b$ .  
 Thus, Susan buys  $6r$  red pens and  $9b$  blue pens.

If Susan bought exactly 31 pens, then  $6r + 9b = 31$ .

Factoring the left side of this equation, we get  $3(2r + 3b) = 31$ .

Since  $r$  and  $b$  are whole numbers, then  $2r + 3b$  is also a whole number, and so the left side of the equation is a multiple of 3.

Since the right side, 31, is not a multiple of 3, there is no solution to the equation  $6r + 9b = 31$  for whole numbers  $r$  and  $b$ .

Therefore, it is not possible for Susan to buy exactly 31 pens.

*Solution 2*

Let the number of packages of red pens bought by Susan be  $r$ , for some whole number  $r$ .  
 Let the number of packages of blue pens bought by Susan be  $b$ , for some whole number  $b$ .  
 Thus, Susan buys  $6r$  red pens and  $9b$  blue pens.

If Susan bought exactly 31 pens, then  $6r + 9b = 31$ .

The smallest possible value for  $b$  is 0, and the largest possible value for  $b$  is 3 since if  $b \geq 4$ , then the number of pens is greater than or equal to  $4 \times 9 = 36$ .

Solving the equation  $6r + 9b = 31$  for  $r$ , we get  $6r = 31 - 9b$ , and so  $r = \frac{31 - 9b}{6}$ .

In the table below, we use this equation to determine the value of  $r$  given that  $b$  can equal 0, 1, 2, or 3.

$b$	$r = \frac{31 - 9b}{6}$
0	$r = \frac{31 - 9(0)}{6} = \frac{31}{6}$
1	$r = \frac{31 - 9(1)}{6} = \frac{22}{6}$
2	$r = \frac{31 - 9(2)}{6} = \frac{13}{6}$
3	$r = \frac{31 - 9(3)}{6} = \frac{4}{6}$

For each of the possible values for  $b$ , the resulting value of  $r$  is not a whole number.

Thus, there is no solution to the equation  $6r + 9b = 31$  for whole numbers  $r$  and  $b$ , and so it is not possible for Susan to buy exactly 31 pens.

2. (a) Expressing  $\frac{1}{5}$  and  $\frac{1}{4}$  with a common denominator of 40, we get  $\frac{1}{5} = \frac{8}{40}$  and  $\frac{1}{4} = \frac{10}{40}$ .  
 We require that  $\frac{n}{40} > \frac{8}{40}$  and  $\frac{n}{40} < \frac{10}{40}$ , thus  $n > 8$  and  $n < 10$ .  
 The only integer  $n$  that satisfies both of these inequalities is  $n = 9$ .
- (b) Expressing  $\frac{m}{8}$  and  $\frac{1}{3}$  with a common denominator of 24, we require  $\frac{3m}{24} > \frac{8}{24}$  and so  $3m > 8$  or  $m > \frac{8}{3}$ .  
 Since  $\frac{8}{3} = 2\frac{2}{3}$  and  $m$  is an integer, then  $m \geq 3$ .  
 Expressing  $\frac{m+1}{8}$  and  $\frac{2}{3}$  with a common denominator of 24, we require  $\frac{3(m+1)}{24} < \frac{16}{24}$  or  $3m + 3 < 16$  or  $3m < 13$ , and so  $m < \frac{13}{3}$ .  
 Since  $\frac{13}{3} = 4\frac{1}{3}$  and  $m$  is an integer, then  $m \leq 4$ .  
 The integer values of  $m$  which satisfy  $m \geq 3$  and  $m \leq 4$  are  $m = 3$  and  $m = 4$ .
- (c) At the start of the weekend, Fiona has played 30 games and has  $w$  wins, so her win ratio is  $\frac{w}{30}$ .  
 Fiona's win ratio at the start of the weekend is greater than  $0.5 = \frac{1}{2}$ , and so  $\frac{w}{30} > \frac{1}{2}$ .  
 Since  $\frac{1}{2} = \frac{15}{30}$ , then we get  $\frac{w}{30} > \frac{15}{30}$ , and so  $w > 15$ .  
 During the weekend Fiona plays five games giving her a total of  $30 + 5 = 35$  games played.  
 Since she wins three of these games, she now has  $w + 3$  wins, and so her win ratio is  $\frac{w+3}{35}$ .  
 Fiona's win ratio at the end of the weekend is less than  $0.7 = \frac{7}{10}$ , and so  $\frac{w+3}{35} < \frac{7}{10}$ .  
 Rewriting this inequality with a common denominator of 70, we get  $\frac{2(w+3)}{70} < \frac{49}{70}$  or  $2(w+3) < 49$  or  $2w + 6 < 49$  or  $2w < 43$ , and so  $w < \frac{43}{2}$ .  
 Since  $\frac{43}{2} = 21\frac{1}{2}$  and  $w$  is an integer, then  $w \leq 21$ .  
 The integer values of  $w$  which satisfy  $w > 15$  and  $w \leq 21$  are  $w = 16, 17, 18, 19, 20, 21$ .
3. (a) Chords  $DE$  and  $FG$  intersect at  $X$ , and so  $(DX)(EX) = (FX)(GX)$  or  $(DX)(8) = (6)(4)$ .  
 Solving this equation for  $DX$ , we get  $DX = \frac{(6)(4)}{8} = \frac{24}{8} = 3$ .  
 The length of  $DX$  is 3.
- (b) Chords  $JK$  and  $LM$  intersect at  $X$ , and so  $(JX)(KX) = (LX)(MX)$  or  $(8y)(10) = (16)(y+9)$ .  
 Solving this equation, we get  $80y = 16y + 144$  or  $64y = 144$ , and so  $y = \frac{144}{64} = \frac{9}{4}$ .
- (c) Chords  $PQ$  and  $ST$  intersect at  $U$ , and so  $(PU)(QU) = (SU)(TU)$ .  
 Since  $TU = TV + UV$ , then  $TU = 6 + n$ .

Substituting values into  $(PU)(QU) = (SU)(TU)$ , we get  $(m)(5) = (3)(6 + n)$ , and so  $5m = 18 + 3n$ .

Chords  $PR$  and  $ST$  intersect at  $V$ , and so  $(PV)(RV) = (TV)(SV)$ .

Since  $SV = SU + UV$ , then  $SV = 3 + n$ .

Substituting values into  $(PV)(RV) = (TV)(SV)$ , we get  $(n)(8) = (6)(3 + n)$ , and so  $8n = 18 + 6n$  or  $2n = 18$  or  $n = 9$ .

Substituting  $n = 9$  into  $5m = 18 + 3n$ , we get  $5m = 18 + 3(9)$  or  $5m = 45$ , and so  $m = 9$ . Therefore,  $m = 9$  and  $n = 9$ .

4. (a) Dave, Yona and Tam have 6, 4 and 8 candies, respectively.

Since they each have an even number of candies, then no candies are discarded.

During Step 2, Dave gives half of his 6 candies to Yona and accepts half of Tam's 8 candies so that he now has  $6 - 3 + 4 = 7$  candies.

Similarly, Yona gives half of her 4 candies to Tam and accepts half of Dave's 6 candies so that she now has  $4 - 2 + 3 = 5$  candies.

Tam gives half of his 8 candies to Dave and accepts half of Yona's 4 candies so that he now has  $8 - 4 + 2 = 6$  candies.

Since Dave has 7 candies, and Yona has 5 candies, they each discard one candy while Tam, who has an even number of candies, does nothing.

These next two steps are summarized in the table to the right.

	Dave	Yona	Tam
Start	3	7	10
After Step 1	2	6	10
After Step 2	6	4	8
After Step 2	7	5	6
After Step 1	6	4	6
After Step 2	6	5	5
After Step 1	6	4	4
After Step 2	5	5	4
After Step 1	4	4	4

Following the given procedure, we continue the table until the procedure ends, as shown.

When the procedure ends, Dave, Yona and Tam each have 4 candies.

- (b) Dave, Yona and Tam begin with 16, 0 and 0 candies, respectively.

The results of each step of the procedure are shown in the table. (We ignore Step 1 when each of the students has an even number of candies.)

Each student has 4 candies when the procedure ends.

	Dave	Yona	Tam
Start	16	0	0
After Step 2	8	8	0
After Step 2	4	8	4
After Step 2	4	6	6
After Step 2	5	5	6
After Step 1	4	4	6
After Step 2	5	4	5
After Step 1	4	4	4

- (c) We begin by investigating the result that Step 2 has on a student's number of candies.

Assume Yona has  $c$  candies, and Dave (from whom Yona receives candies), has  $d$  candies. Further, assume that  $c$  and  $d$  are both even integers.

During Step 2, Yona will give half of her candies away, leaving her with  $\frac{c}{2}$  candies.

In this same Step 2, Yona will also receive  $\frac{d}{2}$  candies from Dave (one half of Dave's  $d$  candies).

Therefore, Yona completes Step 2 with  $\frac{c}{2} + \frac{d}{2} = \frac{c+d}{2}$  candies, which is the average of the  $c$  and  $d$  candies that Yona and Dave respectively began the step with.

On Wednesday, Dave starts with  $2n$  candies, and each of Yona and Tam starts with  $2n + 3$

candies.

Since  $2n + 3$  is 3 more than a multiple of 2, then  $2n + 3$  is an odd integer for any integer  $n$ . That is, we begin the procedure by performing Step 1 which leaves Dave with  $2n$  candies ( $2n$  is even and so no candies are discarded), and each of Yona and Tam with  $2n + 2$  candies.

After Step 2, Yona will have the average of her number of candies,  $2n + 2$ , and Dave's number of candies,  $2n$ , or  $\frac{(2n + 2) + 2n}{2} = \frac{4n + 2}{2} = 2n + 1$ .

Tam will have the average of his number of candies,  $2n + 2$ , and Yona's number of candies,  $2n + 2$ , which is  $2n + 2$ .

Dave will have the average of his number of candies,  $2n$ , and Tam's number of candies,  $2n + 2$ , or  $\frac{(2n + 2) + 2n}{2} = \frac{4n + 2}{2} = 2n + 1$ .

Since Yona and Dave each now have an odd number of candies, Step 1 is performed.

The procedure is continued in the table shown.

	Dave	Yona	Tam
Start	$2n$	$2n + 3$	$2n + 3$
After Step 1	$2n$	$2n + 2$	$2n + 2$
After Step 2	$2n + 1$	$2n + 1$	$2n + 2$
After Step 1	$2n$	$2n$	$2n + 2$
After Step 2	$2n + 1$	$2n$	$2n + 1$
After Step 1	$2n$	$2n$	$2n$

At the end of the procedure, each student has  $2n$  candies.

- (d) On Thursday, Dave begins with  $2^{2017}$  candies, gives one half or  $\frac{1}{2} \times 2^{2017} = 2^{2016}$  to Yona, receives 0 from Tam, and thus completes the first Step 2 having  $2^{2016}$  candies.

In the table shown, we proceed with the first few steps to get a sense of what is happening early in the procedure. (We again ignore Step 1 when each student has an even number of candies.)

	Dave	Yona	Tam
Start	$2^{2017}$	0	0
After Step 2	$2^{2016}$	$2^{2016}$	0
After Step 2	$2^{2015}$	$2^{2016}$	$2^{2015}$
After Step 2	$2^{2015}$	$2^{2014} + 2^{2015}$	$2^{2015} + 2^{2014}$
		$= 2^{2014} + 2 \times 2^{2014}$	$= 2 \times 2^{2014} + 2^{2014}$
		$= 3 \times 2^{2014}$	$= 3 \times 2^{2014}$
After Step 2	$2^{2015} + 2^{2013}$	$2^{2013} + 2^{2015}$	$2^{2015} + 2^{2014}$
	$= 2^2 \times 2^{2013} + 2^{2013}$	$= 2^{2013} + 2^2 \times 2^{2013}$	$= 2 \times 2^{2014} + 2^{2014}$
	$= 5 \times 2^{2013}$	$= 5 \times 2^{2013}$	$= 3 \times 2^{2014}$

As was demonstrated in part (c), each application of Step 2 gives the average number of candies that two students had prior to the step.

If each of the three students has a number of candies that is divisible by  $2^k$  for some positive integer  $k$ , then after performing Step 2, each student will have a number of candies that is divisible by  $2^{k-1}$ . Why?

If Yona has  $a$  candies and Dave has  $b$  candies, where both  $a$  and  $b$  are divisible by  $2^k$ , then after Step 2, Yona's number of candies is the average  $\frac{a + b}{2} = \frac{a}{2} + \frac{b}{2}$ .

Since  $a$  is divisible by  $2^k$ , then  $\frac{a}{2}$  is divisible by  $2^{k-1}$  and similarly  $\frac{b}{2}$  is divisible by  $2^{k-1}$  and so their sum is at least divisible by  $2^{k-1}$  (and possibly more).

We proceed by introducing 4 important facts which will lead us to our conclusion.

Important Fact #1:

We are starting with  $2^{2017}$ , 0 and 0 candies, each of which is divisible by  $2^{2017}$ .

The first application of Step 2 gives three numbers, each of which is divisible by  $2^{2016}$ .

The second application of Step 2 gives three numbers, each of which is divisible by  $2^{2015}$ .

The third application of Step 2 gives three numbers, each of which is divisible by  $2^{2014}$ , and so on. (We can verify this in the table above.)

That is, starting with  $2^{2017}$ , 0 and 0 candies, we are able to apply Step 2 2017 times in a row.

We note that at each of these 2017 steps, the number of candies that each student has is even, and therefore Step 1 is never applied (no candies have been discarded), and so the total number of candies shared by the three students is still  $2^{2017}$ .

Important Fact #2:

If we begin Step 2 with  $2a$ ,  $2a$  and  $2b$  candies (exactly two students having an equal number of candies), then the result after applying Step 2 is  $a + b$ ,  $2a$ , and  $a + b$  candies.

That is, there are still exactly two students who have an equal number of candies.

Important Fact #3:

If we begin Step 2 with  $2a$ ,  $2a$  and  $2b$  candies where  $a < b$ , then we call this a “2 low, 1 high state” (the two equal numbers are less than the third).

Applying Step 2 to  $2a$ ,  $2a$  and  $2b$  (a “2 low, 1 high state”), gives  $a + b$ ,  $2a$ , and  $a + b$  which is a “2 high, 1 low state”. (Since  $a < b$ , then  $a + a < b + a$  or  $2a < a + b$ .)

Similarly, applying Step 2 again to this “2 high, 1 low state” gives a “2 low, 1 high state”. Since we begin with  $2^{2017}$ , 0 and 0 candies, which is a “2 low, 1 high state”, then after 2017 applications of Step 2, we will be at a “2 high, 1 low state”.

Important Fact #4:

Beginning with  $2a$ ,  $2a$  and  $2b$  candies, the positive difference between the high number of candies and the low number of candies is  $2b - 2a$  (or  $2a - 2b$  if  $a > b$ ).

After applying Step 2, we have  $a + b$ ,  $2a$ , and  $a + b$  candies and the positive difference between the high and low numbers of candies is  $b - a$  (or  $a - b$  if  $a > b$ ).

That is, applying Step 2 once decreases the positive difference between the high and low numbers of candies by a factor of 2 (that is,  $a - b = \frac{1}{2}(2a - 2b)$ ).

Therefore, beginning with  $2^{2017}$ , 0 and 0 candies, whose positive difference is  $2^{2017}$ , and applying Step 2 2017 times gives a “2 high, 1 low state” where the positive difference between the high and low numbers is 1.

That is, after applying Step 2 2017 times, the number of candies is  $n + 1$ ,  $n + 1$  and  $n$  for some non-negative integer  $n$ .

Conclusion:

Since we haven't applied Step 1, then there are still  $2^{2017}$  candies shared between the three students.

If  $n$  is odd, then the number of candies,  $3n + 2$ , is odd.

Since  $3n + 2$  is equal to  $2^{2017}$ , this is not possible and so  $n$  is even.

Since  $n$  is even, then  $n + 1$  is odd and so we apply Step 1 to  $n + 1$ ,  $n + 1$  and  $n$  candies so that each student has an equal number of candies,  $n$ .

Two candies were discarded in the application of Step 1 and so there are now  $2^{2017} - 2$  candies remaining.

Since each student has an equal number of candies, and there are  $2^{2017} - 2$  candies in total, the procedure ends with each student having  $\frac{2^{2017} - 2}{3}$  candies.



The CENTRE for EDUCATION  
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## ***2016 Fryer Contest***

**Wednesday, April 13, 2016**  
(in North America and South America)

**Thursday, April 14, 2016**  
(outside of North America and South America)

*Solutions*



1. (a) The total score for School A is found by adding the scores of the four students competing from the school.  
Therefore, the total score for School A is  $12 + 8 + 10 + 6 = 36$ .
  - (b) Since the total score for School A is 36, then the total score for School B is also 36.  
The scores of the four students competing from School B are 17, 5, 7, and  $x$ , and so  $17 + 5 + 7 + x = 36$  or  $29 + x = 36$  and so  $x = 7$ .
  - (c) The score for Student 4 at School C,  $z$ , is twice that of Student 3 at School C,  $y$ .  
Thus,  $z = 2y$ .  
The scores of the four students competing from School C are 9, 15,  $y$ , and  $z$ .  
Since the total score for School C is also 36, and  $z = 2y$ , then  $9 + 15 + y + 2y = 36$  or  $24 + 3y = 36$  or  $3y = 12$ , and so  $y = 4$ .  
Therefore, the score for Student 3 at School C is 4, and the score for Student 4 at School C is  $2(4) = 8$ .
- 
2. (a) Every 2 seconds, Esther takes 5 steps and each of her steps is 0.4 m long.  
Therefore, in 2 seconds Esther travels a distance of  $5 \times 0.4 = 2$  m.
  - (b) *Solution 1*  
Every 2 seconds, Paul takes 5 steps and each of his steps is 1.2 m long.  
Therefore, every 2 seconds Paul travels a distance of  $5 \times 1.2 = 6$  m.  
Since Paul travels 6 m every 2 seconds, his speed is  $6 \div 2 = 3$  m/s.  
  
*Solution 2*  
Paul takes 5 steps every 2 seconds and so Paul takes 2.5 steps every second.  
The length of each of Paul's steps is 1.2 m, and so every second Paul travels a distance of  $2.5 \times 1.2 = 3$  m.  
Thus, Paul travels at a speed of 3 m/s.
  - (c) *Solution 1*  
Paul travels at a speed of 3 m/s, and so in 120 seconds (2 minutes), Paul travels a distance of  $3 \times 120 = 360$  m.  
Esther travels 2 m every 2 seconds and so she travels at a speed of 1 m/s.  
In 120 seconds, Esther travels a distance of  $1 \times 120 = 120$  m.  
If Paul and Esther start a race at the same time, then after 2 minutes, Paul will be  $360 - 120 = 240$  m ahead of Esther.  
  
*Solution 2*  
Each of Paul's steps is  $1.2 - 0.4 = 0.8$  m longer than each of Esther's steps.  
Every 2 seconds, Paul and Esther each take 5 steps, and so in  $2 \times 60 = 120$  seconds (2 minutes), Paul and Esther each take  $5 \times 60 = 300$  steps.  
If Paul and Esther start a race at the same time, then after 2 minutes Paul will be  $300 \times 0.8 = 240$  m ahead of Esther.  
  
*Solution 3*  
Paul travels at a speed of 3 m/s, and Esther travels 2 m every 2 seconds, so she travels at a speed of 1 m/s.  
This means that in 1 second, Paul travels 3 m and Esther travels 1 m.  
Every second, Paul travels  $3 - 1 = 2$  m farther than Esther.  
If Paul and Esther start a race at the same time, then after 120 seconds (2 minutes), Paul will be  $120 \times 2 = 240$  m ahead of Esther.

(d) *Solution 1*

Esther travels 2 m every 2 seconds or 1 m/s, and so in 180 seconds (3 minutes), Esther travels  $180 \times 1 = 180$  m.

Paul travels at a speed of 3 m/s, which is 2 m/s faster than the speed at which Esther travels.

Therefore, every second, Paul travels 2 m farther than Esther travels.

Since Esther begins the race 180 m ahead of Paul, it will take Paul  $180 \div 2 = 90$  s to catch Esther.

*Solution 2*

Esther travels 2 m every 2 seconds or 1 m/s, and so in 180 seconds (3 minutes), Esther travels  $180 \times 1 = 180$  m.

Each of Paul's steps is  $1.2 - 0.4 = 0.8$  m longer than each of Esther's steps.

Since Paul and Esther each step at the same rate (5 steps every 2 seconds), then it will take Paul  $180 \div 0.8 = 225$  steps to catch Esther.

Paul takes 5 steps every 2 seconds, and so it will take Paul  $\frac{225}{5} \times 2 = 45 \times 2 = 90$  s to catch Esther.

3. (a) *Solution 1*

In  $\triangle ABC$ ,  $AD$  is a median and so  $D$  is the midpoint of  $BC$ .

Since  $BC = 12$  and  $D$  is the midpoint of  $BC$ , then  $CD = \frac{12}{2} = 6$ .

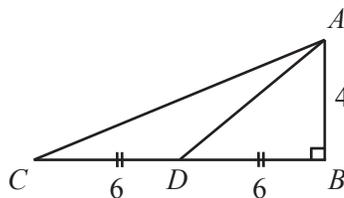
In  $\triangle ACD$ , base  $CD$  has length 6, and corresponding height  $AB$  has length 4. (Since  $\angle ABC = 90^\circ$ ,  $AB$  is the height of  $\triangle ACD$  even though  $AB$  is outside  $\triangle ACD$ .)

Thus,  $\triangle ACD$  has area  $\frac{1}{2}(6)(4) = 12$ .

*Solution 2*

In  $\triangle ABC$ ,  $AD$  is a median and so  $D$  is the midpoint of  $BC$ .

Since  $BC = 12$  and  $D$  is the midpoint of  $BC$ , then  $CD = DB = 6$ .



In  $\triangle ABD$ ,  $AB = 4$ ,  $DB = 6$ , and  $\angle ABD = 90^\circ$ , and so  $\triangle ABD$  has area  $\frac{1}{2}(6)(4) = 12$ .

Similarly,  $\triangle ABC$  has area  $\frac{1}{2}(12)(4) = 24$ , and so the area of  $\triangle ACD$  is the area of  $\triangle ABC$  minus the area of  $\triangle ABD$ , or  $24 - 12 = 12$ .

*Solution 3*

In  $\triangle ABC$ ,  $AB = 4$ ,  $BC = 12$ , and  $\angle ABC = 90^\circ$ , and so  $\triangle ABC$  has area  $\frac{1}{2}(12)(4) = 24$ . A median of  $\triangle ABC$  divides the triangle into two equal areas. Why?

In  $\triangle ABC$ ,  $AD$  is a median and so  $D$  is the midpoint of  $BC$ .

Therefore,  $\triangle ACD$  and  $\triangle ABD$  have equal bases ( $CD = BD$ ).

Further, the height of  $\triangle ABD$  is equal to the height of  $\triangle ACD$  (both are  $AB$ ).

Thus,  $\triangle ACD$  and  $\triangle ABD$  have equal bases and equal heights.

Since the area of each triangle equals one-half times the base times the height, then  $\triangle ABD$  and  $\triangle ACD$  have equal areas and so median  $AD$  divides  $\triangle ABC$  into equal areas.

Since  $\triangle ABC$  has area 24, then  $\triangle ACD$  has area  $\frac{24}{2} = 12$ .

(b) *Solution 1*

In  $\triangle FSG$ ,  $FS = 18$ ,  $SG = 24$ , and  $\angle FSG = 90^\circ$ .

Thus, by the Pythagorean Theorem,  $FG = \sqrt{18^2 + 24^2} = \sqrt{324 + 576} = \sqrt{900} = 30$  (since  $FG > 0$ ).

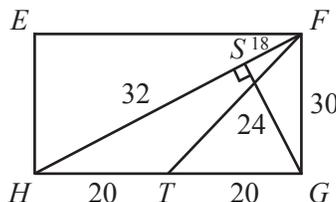
Since  $S$  is on  $FH$  so that  $FS = 18$  and  $SH = 32$ , then  $FH = FS + SH = 18 + 32 = 50$ .

In  $\triangle FGH$ ,  $FH = 50$ ,  $FG = 30$ , and  $\angle FGH = 90^\circ$ .

Thus, by the Pythagorean Theorem,  $GH = \sqrt{50^2 - 30^2} = \sqrt{2500 - 900} = \sqrt{1600} = 40$  (since  $GH > 0$ ).

In  $\triangle FGH$ ,  $FT$  is a median and so  $T$  is the midpoint of  $GH$ .

In  $\triangle FHT$ , base  $HT = \frac{40}{2} = 20$ , and height  $FG = 30$ . (Since  $\angle FGH = 90^\circ$ ,  $FG$  is the height of  $\triangle FHT$  even though  $FG$  is outside  $\triangle FHT$ .)



Thus,  $\triangle FHT$  has area  $\frac{1}{2}(20)(30) = 300$ .

*Solution 2*

Since  $S$  is on  $FH$  so that  $FS = 18$  and  $SH = 32$ , then  $FH = FS + SH = 18 + 32 = 50$ .

In  $\triangle FGH$ , base  $FH = 50$ , and height  $SG = 24$  (since  $SG$  is perpendicular to  $FH$ ,  $SG$  is a height of  $\triangle FGH$ ).

Thus,  $\triangle FGH$  has area  $\frac{1}{2}(50)(24) = 600$ .

The median of a triangle divides the area of the triangle in half.

(Solution 3 to (a) shows an example of why a median divides a triangle's area in half.)

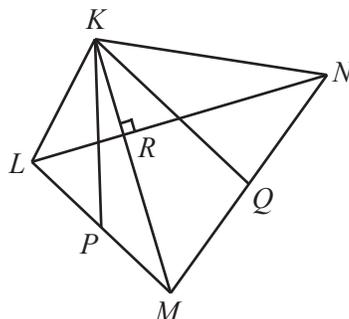
Since  $FT$  is a median of  $\triangle FGH$ , then the area of  $\triangle FHT = \frac{600}{2} = 300$ .

(c) We use the notation  $|\triangle KLM|$  to represent the area of  $\triangle KLM$ ,  $|KPMQ|$  to represent the area of  $KPMQ$ , and so on.

In  $\triangle KLM$ ,  $KP$  is a median and so  $2|\triangle KPM| = |\triangle KLM|$ .

(Solution 3 to (a) shows an example of why a median divides a triangle's area in half.)

In  $\triangle KMN$ ,  $KQ$  is a median and so  $2|\triangle KMQ| = |\triangle KMN|$ .



Therefore,

$$|KLMN| = |\triangle KLM| + |\triangle KMN| = 2|\triangle KPM| + 2|\triangle KMQ|$$

and

$$|KPMQ| = |\triangle KPM| + |\triangle KMQ|$$

which tells us that  $|KLMN| = 2|KPMQ|$ .

Since  $|KPMQ| = 63$ , then  $|KLMN| = 2|KPMQ| = 2(63) = 126$ .

Now  $|KLMN| = |\triangle KRL| + |\triangle LRM| + |\triangle KRN| + |\triangle NRM|$ .

Each of these four triangles is right-angled.

Since  $KR = x$  and  $LR = 6$ , then  $|\triangle KRL| = \frac{1}{2}x(6) = 3x$ .

Since  $LR = 6$  and  $RM = 2x + 2$ , then  $|\triangle LRM| = \frac{1}{2}(6)(2x + 2) = 6x + 6$ .

Since  $KR = x$  and  $RN = 12$ , then  $|\triangle KRN| = \frac{1}{2}x(12) = 6x$ .

Since  $RN = 12$  and  $RM = 2x + 2$ , then  $|\triangle NRM| = \frac{1}{2}(12)(2x + 2) = 12x + 12$ .

Therefore,  $126 = 3x + (6x + 6) + 6x + (12x + 12)$  or  $126 = 27x + 18$  or  $27x = 108$  and so  $x = 4$ .

4. (a) Because the entries from one column do not affect the possible entries in another column, then the smallest possible sum of the numbers in a row equals the sum of the smallest possible number in each column.

Column	Possible entries	Smallest possible entry
B	1, 2, 3, ..., 13, 14, 15	1
I	16, 17, 18, ..., 28, 29, 30	16
N	0, 31, 32, 33, ..., 43, 44, 45	0
G	46, 47, 48, ..., 58, 59, 60	46
O	61, 62, 63, ..., 73, 74, 75	61

Thus, the smallest possible sum of the numbers in a row on a BINGO card is equal to  $1 + 16 + 0 + 46 + 61$  or 124. (This sum can only occur in the middle row, since the entry 0 can only occur in the middle of the square.)

- (b) *Solution 1*

From part (a), the smallest possible sum of the numbers in a row is 124.

This minimum row sum occurs in the  $3^{\text{rd}}$  row so that the number in column N is 0.

(Note that if we do not use the  $3^{\text{rd}}$  row, then the smallest number that can occur in column N is 31, and thus the minimum possible row sum in any row other than the  $3^{\text{rd}}$  is  $1 + 16 + 31 + 46 + 61 = 124 + 31 = 155$ .)

The minimum sum of the numbers in a diagonal is also 124 since the smallest possible number in each column (including the middle entry 0) may be used in a diagonal sum.

The minimum row sum and the minimum diagonal sum each use the numbers 1, 16, 0, 46, and 61.

However, the number 1 cannot occur in both the  $3^{\text{rd}}$  row and in the diagonal since every BINGO card is filled with twenty-five different integers.

The smallest two numbers that can appear in the  $3^{\text{rd}}$  row and in the diagonal in column B are 1 and 2.

Similarly, the smallest two numbers that can appear in the  $3^{\text{rd}}$  row and in the diagonal in column I are 16 and 17.

In column N, the  $3^{\text{rd}}$  row and the diagonal intersect and share the smallest number 0.

In column G, the number 46 cannot appear in both the  $3^{\text{rd}}$  row and in the diagonal, and so the smallest two numbers that can appear in the  $3^{\text{rd}}$  row and in the diagonal in column G are 46 and 47.

Similarly, the smallest two numbers that can appear in the  $3^{\text{rd}}$  row and in the diagonal in column O are 61 and 62.

Therefore, in any BINGO card, the combined list of numbers in a diagonal sum and a row sum must include 10 numbers that are at least as large as 1, 2, 16, 17, 0, 0, 46, 47, 61, 62.

The sum of these numbers is  $1 + 2 + 16 + 17 + 0 + 0 + 46 + 47 + 61 + 62 = 252$ .

Thus, if Carrie's BINGO card has a row and a diagonal each with the same sum, then this sum must be at least one-half of this total; that is, the minimum such sum is  $\frac{1}{2}(252) = 126$ . One BINGO card showing this minimum equal row and diagonal sum of 126 is possible, is shown below.

B	I	N	G	O
1				
	17			
2	16	0	47	61
			46	
				62

We confirm that the row and diagonal sums are  $2+16+0+47+61 = 1+17+0+46+62 = 126$ , as claimed.

Each of the empty spaces in the card may be filled with any of the appropriate numbers not yet used.

*Solution 2*

From part (a), the smallest possible sum of the numbers in a row is 124.

This minimum row sum occurs in the  $3^{rd}$  row so that the number in column N is 0.

(Note that if we do not use the  $3^{rd}$  row, then the smallest number that can occur in column N is 31, and thus the minimum possible row sum in any row other than the  $3^{rd}$  is  $1 + 16 + 31 + 46 + 61 = 124 + 31 = 155$ .)

The minimum sum of the numbers in a diagonal is also 124 since the smallest possible number in each column (including the middle entry 0) could be used in a diagonal sum.

The minimum row sum and the minimum diagonal sum each must use the numbers 1, 16, 0, 46, 61.

However, the number 1 cannot occur in both the  $3^{rd}$  row and in a diagonal, since 1 cannot occur twice in the B column.

Similarly, 16 cannot occur in both the  $3^{rd}$  row and in the diagonal, since 16 cannot occur twice in the I column. As well, 46 and 61 cannot occur in both the  $3^{rd}$  row and in a diagonal. Therefore, it is not possible for a BINGO card to have both the  $3^{rd}$  row and a diagonal sum to 124.

A row sum of 126 and a diagonal sum of 126 is possible, as the following BINGO card shows:

B	I	N	G	O
1				
	16			
2	17	0	46	61
			47	
				62

We confirm that the row and diagonal sums are  $2 + 17 + 0 + 46 + 61 = 126$  and  $1 + 16 + 0 + 47 + 62 = 126$ , as claimed. Each of the empty spaces in the card may be filled with any of the appropriate numbers not yet used to create a full BINGO card.

Is it possible to have a BINGO card with a row sum of 125 and a diagonal sum of 125?

If not, then the smallest possible number that can be both a row sum and a diagonal sum will be 126.

Consider the smallest possible diagonal sum  $1 + 16 + 0 + 46 + 61 = 124$ .

Since 1, 16, 0, 46, 61 are the smallest possible entries in each column, then a diagonal sum of 125 can only be created by replacing exactly one of the four integers 1, 16, 46, 61 by the integer that is one larger.

In particular, the possible diagonals with a sum of 125 are

$$2 + 16 + 0 + 46 + 61 = 125 \quad 1 + 17 + 0 + 46 + 61 = 125$$

$$1 + 16 + 0 + 47 + 61 = 125 \quad 1 + 16 + 0 + 46 + 62 = 125$$

For the same reason, these are also the possible  $3^{\text{rd}}$  rows with a sum of 125.

It is not possible for a BINGO card to have one of these sums as a diagonal sum and a different one of these sums as a  $3^{\text{rd}}$  row sum, since each pair of these sums has more numbers than just the 0 in common. (For example,  $2+16+0+46+61$  and  $1+16+0+47+61$  share 16, 0 and 61, and the number 16 cannot appear in the I column in both a diagonal and the  $3^{\text{rd}}$  row.)

Therefore, the smallest possible number that can be both a diagonal sum and a row sum is 126.

(c) *Solution 1*

The maximum possible sum of the numbers in the  $3^{\text{rd}}$  row and in the diagonal is  $15 + 30 + 0 + 60 + 75 = 180$ .

We need the sum of the numbers in the  $3^{\text{rd}}$  row and the sum of the numbers in the diagonal to both be 177.

We determine the number of ways in which this can be done by starting with the largest possible numbers and reducing these numbers to reduce the sums to 177.

Thus, we call the missing numbers in the  $3^{\text{rd}}$  row  $15 - W, 30 - X, 60 - Y, 75 - Z$  for some integers  $W, X, Y, Z$  and the missing numbers in the diagonal  $15 - w, 30 - x, 60 - y, 75 - z$  for some integers  $w, x, y, z$ , as shown:

B	I	N	G	O
$15 - w$	23	35	47	65
5	$30 - x$	31	52	63
$15 - W$	$30 - X$	0	$60 - Y$	$75 - Z$
11	20	40	$60 - y$	69
9	18	38	48	$75 - z$

Since the numbers in the first column are between 1 and 15, inclusive, then  $w \geq 0$  and  $W \geq 0$ . Similarly, each of  $x, X, y, Y, z, Z$  are greater than or equal to 0.

Since the numbers in the  $3^{\text{rd}}$  row have a sum of 177, then

$$(15 - W) + (30 - X) + 0 + (60 - Y) + (75 - Z) = 177$$

or  $180 - (W + X + Y + Z) = 177$  and so  $W + X + Y + Z = 3$ .

Similarly, we have  $w + x + y + z = 3$ .

Since the B column cannot contain repeated numbers, then  $15 - w$  and  $15 - W$  cannot be equal, which means that we cannot have  $w = W$ . Also,  $x \neq X$  and  $y \neq Y$  and  $z \neq Z$ .

The number of BINGO cards with the desired property is equal to the number of ways that we can choose non-negative integers  $w, x, y, z, W, X, Y, Z$  with the correct sums and so that no two numbers in the same column are equal.

Since  $W, X, Y, Z$  are integers that are at least 0, then they must be 3, 0, 0, 0 in some order or 2, 1, 0, 0 in some order or 1, 1, 1, 0 in some order:

For the sum of non-negative integers to be 3, no single integer can be larger than 3. If one integer is 3, the rest are 0. If one integer is 2, then we must have one 1 and the rest equal to 0. If no integers equal 3 or 2, then we must have three 1s.

Similarly,  $w, x, y, z$  must be 3, 0, 0, 0 in some order or 2, 1, 0, 0 in some order or 1, 1, 1, 0 in some order.

Note that since none of the values can be larger than 3, then the missing entries in the B, I, G, and O columns are at least 12, 27, 57, and 72, respectively, so cannot duplicate existing entries.

To count the BINGO cards, we now count the possible combinations of values for  $W, X, Y, Z$  and  $w, x, y, z$ .

In this discussion, we call  $w$  and  $W$  *corresponding positions*. Similarly,  $x$  and  $X$ ,  $y$  and  $Y$ , and  $z$  and  $Z$  will be called corresponding positions.

Case 1:  $W, X, Y, Z$  are 3, 0, 0, 0 in some order

- (i)  $w, x, y, z$  cannot be 3, 0, 0, 0. If it were, then at least two pairs of corresponding positions will equal 0, which would mean that at least two columns of the card would contain the same number twice. (For example, if  $W = 3, X = 0, Y = 0, Z = 0$  and  $w = 0, x = 0, y = 3, z = 0$ , then  $X = x = 0$  and  $Z = z = 0$  which means that the I column will include 30 twice and the O column will include 75 twice.)
- (ii)  $w, x, y, z$  cannot be 2, 1, 0, 0 because at least one pair of corresponding positions will equal 0, which means that at least one column of the BINGO card will contain the same number twice.
- (iii)  $w, x, y, z$  could be 1, 1, 1, 0 in some order. In how many ways can this happen?

Since  $W, X, Y, Z$  are 3, 0, 0, 0 in some order, then there are 4 possible positions in which the 3 can go. The remaining three positions must be 0.

Looking at  $w, x, y, z$ , the 0 must go in the position corresponding to the 3 (since there cannot be two 0s in corresponding positions) and so the 1s go in the remaining positions.

In total, this means that there are 4 possible ways in which this can happen.

Case 2:  $W, X, Y, Z$  are 2, 1, 0, 0 in some order

- (i)  $w, x, y, z$  cannot be 3, 0, 0, 0 as we saw in Case 1(ii).
- (ii)  $w, x, y, z$  could be 2, 1, 0, 0 in some order.  
Looking at  $W, X, Y, Z$ , there are 4 possible positions for the 2. For each of these positions, there are 3 possible positions for the 1. The 0s go in the remaining two positions.  
Looking at  $w, x, y, z$ , the 0s must go in the positions that correspond to the 2 and 1 among  $W, X, Y, Z$ . This means that there are 2 possible positions for the 2 and then the 1 is placed in the last position.  
Overall, there are  $4 \cdot 3 \cdot 2 = 24$  ways in which this can be done.
- (iii)  $w, x, y, z$  could be 1, 1, 1, 0 in some order.  
In how many ways can this happen?

Looking at  $W, X, Y, Z$ , there are 4 possible positions for the 2. For each of these positions, there are 3 possible positions for the 1. The 0s go in the remaining two positions.

Looking at  $w, x, y, z$ , the 0 must go in the corresponding position to the 1 among  $W, X, Y, Z$ , since there cannot be two 1s in this position. The positions of the 1s are then completely determined.

Overall, there are  $4 \cdot 3 = 12$  ways in which this can be done.

Case 3:  $W, X, Y, Z$  are 1, 1, 1, 0 in some order

- (i)  $w, x, y, z$  could be 3, 0, 0, 0 in some order. As we saw in Case 1(iii), there are 4 ways in which this can happen.
- (ii)  $w, x, y, z$  could be 2, 1, 0, 0 in some order. As we saw in Case 2(iii), there are 12 ways in which this can happen.
- (iii)  $w, x, y, z$  cannot be 1, 1, 1, 0 in some order because at least two pairs of corresponding variables will equal 1.

In total, there are thus  $4 + 24 + 12 + 4 + 12 = 56$  ways in which  $W, X, Y, Z, w, x, y, z$  can be determined.

Each set of values of these variables gives a BINGO card with the desired property, and so there are 56 ways of completing the BINGO card so that the sum of the numbers in the diagonal and in the 3<sup>rd</sup> row are each 177.

### *Solution 2*

The maximum possible sum of the numbers in the 3<sup>rd</sup> row and in the diagonal is  $15 + 30 + 0 + 60 + 75 = 180$ .

We require both the sum of the numbers in the 3<sup>rd</sup> row and the sum of the numbers in the diagonal to be 177.

Since 177 is 3 less than the maximum possible sum of 180, then any of the missing numbers in the given BINGO card can be at most 3 less than the largest number that can appear in columns B, I, G, and O (column N is fixed at 0).

That is, the smallest number that can appear in the 3<sup>rd</sup> row and in the diagonal of column B is  $15 - 3 = 12$ .

Similarly, the smallest numbers that can appear in the 3<sup>rd</sup> row and in the diagonal of columns I, G and O are 27, 57 and 72, respectively.

Thus, the missing numbers in the given BINGO card must be chosen from:

12, 13, 14, 15 in column B,  
27, 28, 29, 30 in column I,  
57, 58, 59, 60 in column G, and  
72, 73, 74, 75 in column O.

(Note that these numbers do not already appear in the given BINGO card and so they each may be chosen to fill blank spaces.)

There are three different methods in which the maximum sum, 180, can be decreased by exactly 3 to give a row or diagonal sum of 177.

From the lists above, we may choose:

- the smallest number from one of the four columns (this number is 3 less than the largest), and choose the largest number from each of the remaining three columns. For example we could choose the smallest number from column B, 12, and the largest numbers from the remaining columns, 30, 60, 75, since  $12 + 30 + 60 + 75 = 177$ , or
- the largest number from one of the four columns, and choose the second largest number from each of the remaining three columns. For example we could choose the largest number from column B, 15, and the second largest numbers from the remaining columns, 29, 59, 74, since  $15 + 29 + 59 + 74 = 177$ , or
- the largest numbers from two of the four columns, and choose the second largest number from one of the remaining two columns and the third largest number from the final column. For example we could choose the largest numbers from columns B

and I, 15 and 30, and the second largest number from column G, 59, and the third largest number from column O, 73, since  $15 + 30 + 59 + 73 = 177$ .

These are the only three methods in which we can decrease the maximum row and diagonal sum of 180 by exactly 3 to give a row or diagonal sum of 177.

We restate these three methods by considering the following table:

	B	I	G	O
<i>P</i>	15	30	60	75
<i>Q</i>	14	29	59	74
<i>R</i>	13	28	28	73
<i>S</i>	12	27	57	72

The three methods for achieving a row or diagonal sum of 177 are:

- (1) choose 1 number from group *S* (the numbers 12, 27, 57, 72), and 3 numbers from group *P*, or
- (2) choose 1 number from group *P*, and 3 numbers from group *Q*, or
- (3) choose 2 numbers from group *P*, 1 number from group *Q*, and 1 number from group *R*.

We require both the  $3^{rd}$  row sum and the diagonal sum to be 177.

Thus, we must use two of the above methods simultaneously and we must ensure that no number appears twice in any given column.

Which pairs of combinations may be chosen from the three methods listed?

If we fill the blanks in the  $3^{rd}$  row of the BINGO card using method 1, then we cannot fill the blanks of the diagonal using method 1 since each application of method 1 requires that we use 3 different numbers from group *P*, and there are only 4 numbers to choose from in any of the groups.

Further, if we fill the blanks in the  $3^{rd}$  row of the BINGO card using method 1, then we cannot fill the blanks of the diagonal using method 3 since this would require 5 numbers from group *P*.

Therefore, if the  $3^{rd}$  row is filled using method 1, then the diagonal must be filled using method 2.

Similarly, we cannot fill the blanks in the  $3^{rd}$  row of the BINGO card using method 2 and at the same time use method 2 to fill the blanks in the diagonal.

We also cannot fill the blanks in the  $3^{rd}$  row of the BINGO card using method 3 and at the same time use method 1 to fill the blanks in the diagonal.

All other combinations of methods are possible and they are summarized in the table:

Method used to fill the $3^{rd}$ row	Method used to fill the diagonal
1: <i>SPPP</i>	2: <i>PQQQ</i>
2: <i>PQQQ</i>	1: <i>SPPP</i>
2: <i>PQQQ</i>	3: <i>PPQR</i>
3: <i>PPQR</i>	2: <i>PQQQ</i>
3: <i>PPQR</i>	3: <i>PPQR</i>

Finally, to count the number of ways to complete the BINGO card, we must count the number of ways to choose numbers that satisfy each of the five combinations listed above.

First Combination: *SPPP* in the  $3^{rd}$  row and *PQQQ* in the diagonal

*SPPP* can occur in 4 ways in the  $3^{rd}$  row:

Select one of the 4 columns (B, I, G, O) in which to choose the group *S* number, and the remaining 3 columns are each filled with their group *P* number.

To complete the diagonal in this same BINGO card using  $PQQQ$ , we recognize that the group  $P$  number must occur in the same column in which the group  $S$  number occurred in the  $3^{rd}$  row, because each of the other 3 group  $P$  numbers are already in the  $3^{rd}$  row.

That is, there is only 1 choice for the placement of the group  $P$  number, and each of the remaining 3 columns in the diagonal will be filled with their group  $Q$  number.

So there are 4 ways to select the numbers for the  $3^{rd}$  row and then only 1 way to select the numbers for the diagonal, and thus there are  $4 \times 1 = 4$  ways to complete the BINGO card using this first combination.

Second Combination:  $PQQQ$  in the  $3^{rd}$  row and  $SPPP$  in the diagonal

The counting here is identical to that of the first combination above, with the roles of the  $3^{rd}$  row and diagonal reversed.

Thus, there are 4 ways in which the BINGO card can be completed using this second combination.

Third Combination:  $PQQQ$  in the  $3^{rd}$  row and  $PPQR$  in the diagonal

$PQQQ$  can occur in 4 ways in the  $3^{rd}$  row:

Select one of the 4 columns (B, I, G, O) in which to choose the group  $P$  number, and the remaining 3 columns are each filled with their group  $Q$  number.

To complete the diagonal in this same BINGO card using  $PPQR$ , we recognize that the group  $Q$  number must occur in the same column in which the group  $P$  number occurred in the  $3^{rd}$  row, because each of the other 3 group  $Q$  numbers are already in the  $3^{rd}$  row.

Next we must fill the remaining 3 columns with their respective  $PPR$  numbers.

These can be ordered in 3 different ways ( $PPR$ ,  $PRP$ , and  $RPP$ ) and so there are 3 ways to fill the remaining 3 columns in the diagonal.

So there are 4 ways to select the numbers for the  $3^{rd}$  row and then  $1 \times 3$  ways to select the numbers for the diagonal, and thus there are  $4 \times 3 = 12$  ways to complete the BINGO card using this third combination.

Fourth Combination:  $PPQR$  in the  $3^{rd}$  row and  $PQQQ$  in the diagonal

The counting here is identical to that of the third combination above, with the roles of the  $3^{rd}$  row and diagonal reversed.

Thus there are 12 ways in which the BINGO card can be completed using this fourth combination.

Fifth Combination:  $PPQR$  in the  $3^{rd}$  row and  $PPQR$  in the diagonal

$PPQR$  can occur in 12 ways in the  $3^{rd}$  row:

Select one of the four columns in which to place the group  $Q$  number. There are 4 choices for this column, and for each of these choices, there are 3 choices of column in which to place the group  $R$  number. The group  $P$  numbers are then placed in the empty columns.

We then need to place the numbers on the diagonal. There are 2 ways to do this:

The group  $P$  numbers on the diagonal must go in the columns corresponding to the locations of the group  $Q$  and  $R$  numbers in the  $3^{rd}$  row.

The group  $Q$  and  $R$  numbers on the diagonal can be placed in the two remaining columns in 2 ways – either  $QR$  or  $RQ$  when reading from left to right.

Therefore, there are  $12 \times 2 = 24$  ways in which the BINGO card can be completed using this fifth combination.

In total, the number of ways to complete the BINGO card so that the sum of the numbers in the diagonal is 177, and the sum of the numbers in the  $3^{rd}$  row is 177 is  $4+4+12+12+24$ , which equals 56.



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## ***2015 Fryer Contest***

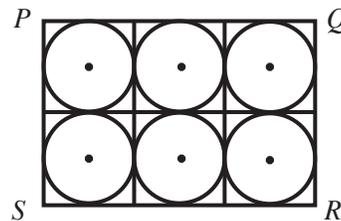
**Thursday, April 16, 2015**  
(in North America and South America)

**Friday, April 17, 2015**  
(outside of North America and South America)

*Solutions*

1. (a) A Model A cylinder has radius  $r = 10$  cm and height  $h = 16$  cm and thus has volume  $V = \pi(10)^2(16) = 1600\pi$  cm<sup>3</sup>.
- (b) A Model B cylinder has radius  $r = 8$  cm and thus has volume  $V = \pi(8)^2(h) = 64\pi h$  cm<sup>3</sup>. Since the volume of a Model B cylinder is equal to the volume of a Model A cylinder, then  $64\pi h = 1600\pi$  and so  $h = \frac{1600\pi}{64\pi} = 25$  cm.  
The height of a Model B cylinder is 25 cm.

- (c) Consider the top-down view of the open Box A shown.  
We label the rectangle as  $PQRS$ .



Since each circle has radius 10 cm, then each circle has diameter 20 cm, and so can be enclosed in a square with side length 20 cm whose sides are parallel to the sides of rectangle  $PQRS$ .

Each circle touches each of the four sides of its enclosing square.

Because each of the circles touches one or two sides of rectangle  $PQRS$  and each of the circles touches two or three of the other circles, then these six squares will fit together without overlapping to completely cover rectangle  $PQRS$ .

Therefore,  $PQ = SR = 3 \times 20 = 60$  cm and  $PS = QR = 2 \times 20 = 40$  cm since  $PQRS$  is three squares wide and two squares tall.

Finally, since the height of each Model A cylinder is 16 cm, then the height of the box is 16 cm, and so the volume of Box A is  $60 \times 40 \times 16 = 38\,400$  cm<sup>3</sup>.

- (d) The volume of Box B is equal to the volume of Box A.  
Intuitively, this makes some sense since the volume of a Model B cylinder is equal to the volume of a Model A cylinder and each box is tightly packed with six cylinders.  
We may follow the approach used in part (c) to demonstrate that the volume of Box B is also 38 400 cm<sup>3</sup>.

Recall from part (b) that a Model B cylinder has a radius of 8 cm and a height of 25 cm. The length of Box B is six times the radius of a Model B cylinder, or  $6 \times 8 = 48$  cm. The width of Box B is four times the radius of a Model B cylinder, or  $4 \times 8 = 32$  cm. The height of Box B equals the height of a Model B cylinder, or 25 cm. Therefore, the volume of Box B is  $48 \times 32 \times 25 = 38\,400$  cm<sup>3</sup> which is equal to the volume of Box A.

2. (a) A quarter is worth \$0.25 and so 3 quarters are worth  $3 \times \$0.25 = \$0.75$ .  
A dime is worth \$0.10 and so 18 dimes are worth  $18 \times \$0.10 = \$1.80$ .  
A nickel is worth \$0.05 and so 25 nickels are worth  $25 \times \$0.05 = \$1.25$ .  
The total value of Susan's coins is  $\$0.75 + \$1.80 + \$1.25 = \$3.80$ .

- (b) *Solution 1*

Pair each of Allen's nickels with a dime.

Since there are an equal number of nickels and dimes (and no other coins), then every nickel pairs with a dime with no coins left over.

Each nickel + dime pairing has a value of  $\$0.05 + \$0.10 = \$0.15$ .

Since Allen's coins have a total value of \$1.50, then he has  $\$1.50 \div \$0.15 = 10$  nickel + dime pairings.

Since there is one nickel in each nickel + dime pairing, then Allen must have 10 nickels.

*Solution 2*

Let the number of nickels that Allen has be  $y$ .

Allen has equal numbers of nickels and dimes and so the number of dimes Allen has is

also  $y$ .

The value of  $y$  nickels is  $\$0.05y$ .

The value of  $y$  dimes is  $\$0.10y$ .

Since the total value of Allen's coins is  $\$1.50$ , then  $0.05y + 0.10y = 1.50$  or  $0.15y = 1.50$

and so  $y = \frac{1.50}{0.15} = 10$ .

Allen has 10 nickels.

(c) A quarter is worth  $\$0.25$  and so  $x$  quarters are worth  $\$0.25x$ .

A dime is worth  $\$0.10$  and so  $2x + 3$  dimes are worth  $\$0.10(2x + 3)$ .

Since Elise has  $\$10.65$  in quarters and dimes, then  $0.25x + 0.10(2x + 3) = 10.65$

or  $0.25x + 0.20x + 0.30 = 10.65$  or  $0.45x = 10.35$  and so  $x = \frac{10.35}{0.45} = 23$ .

3. (a) Using the formula given, the sum of the first 200 positive integers

$$1 + 2 + 3 + \cdots + 198 + 199 + 200 = \frac{200(200 + 1)}{2} = 100(201) = 20\,100.$$

(b) The sum of the first 200 positive integers is equal to the sum of the first 150 positive integers added to the sum of the 50 consecutive integers  $151 + 152 + 153 + \cdots + 198 + 199 + 200$ . That is,

$$1 + 2 + \cdots + 198 + 199 + 200 = (1 + 2 + \cdots + 148 + 149 + 150) + (151 + 152 + \cdots + 198 + 199 + 200).$$

Therefore, the sum of the 50 consecutive integers  $151 + 152 + 153 + \cdots + 198 + 199 + 200$  is equal to the difference between the sum of the first 200 positive integers and the sum of the first 150 positive integers.

From part (a), the sum of the first 200 positive integers is 20 100.

Using the formula, the sum of the first 150 positive integers

$$1 + 2 + 3 + \cdots + 148 + 149 + 150 = \frac{150(150 + 1)}{2} = 75(151) = 11\,325.$$

Therefore, the sum of the 50 consecutive integers beginning at 151,

$$\begin{aligned} 151 + 152 + \cdots + 199 + 200 &= (1 + 2 + \cdots + 199 + 200) - (1 + 2 + \cdots + 149 + 150) \\ &= 20\,100 - 11\,325 \\ &= 8\,775. \end{aligned}$$

(c) Let the required sum,  $1 + 2 + 4 + 5 + 7 + 8 + 10 + 11 + \cdots + 998 + 1000$ , be  $S$ .

Let the sum of the first 333 positive multiples of 3,  $3 + 6 + 9 + 12 + \cdots + 996 + 999$ , be  $T$ .

The sum of the first 1000 positive integers is equal to  $S + T$ .

That is,

$$1 + 2 + \cdots + 998 + 999 + 1000 = (1 + 2 + 4 + 5 + 7 + \cdots + 998 + 1000) + (3 + 6 + \cdots + 996 + 999).$$

Therefore, the required sum  $S$  is equal to the difference between the sum of the first 1000 positive integers and  $T$ .

Using the formula, the sum of the first 1000 positive integers

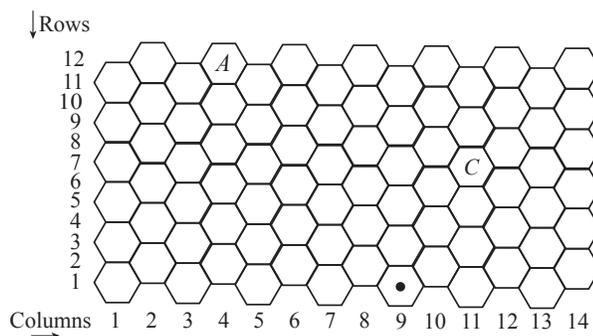
$$1 + 2 + 3 + \cdots + 998 + 999 + 1000 = \frac{1000(1000 + 1)}{2} = 500(1001) = 500\,500.$$

We may determine the sum  $T$  by first removing the common factor 3 from each term and then using the formula.

$$\begin{aligned}
 3 + 6 + 9 + \cdots + 993 + 996 + 999 &= 3(1 + 2 + 3 + \cdots + 331 + 332 + 333) \\
 &= 3 \left( \frac{333(333 + 1)}{2} \right) && \text{(by the formula)} \\
 &= 3(333 \times 167) \\
 &= 166\,833.
 \end{aligned}$$

Finally, the required sum  $S$  is  $500\,500 - 166\,833 = 333\,667$ .

4. We number the rows and columns of hexagons as shown:



Notice that column 1 includes hexagons only in the odd-numbered rows, column 2 includes hexagons only in the even-numbered rows, and so on.

Therefore, the token starts in the hexagon in row 1, column 9 (we write r1c9), the hexagon labelled  $A$  is in row 12, column 4 (r12c4), and the hexagon labelled  $C$  is in row 7, column 11 (r7c11).

A  $\uparrow$  step increases the row number by 2 and does not change the column number.

A  $\swarrow$  step increases the row number by 1 and decreases the column number by 1.

A  $\nearrow$  step increases the row number by 1 and increases the column number by 1.

(a) We must move the token from r1c9 to r12c4 in as few steps as possible.

At least 5  $\swarrow$  steps are needed in order to move the token 5 columns to the left. (More  $\swarrow$  steps could be made if they were balanced by  $\nearrow$  steps.)

One way to move the token from r1c9 to r12c4 is to use 5  $\swarrow$  steps (taking the token to r6c4 since each  $\swarrow$  step moves the token 1 column to the left and 1 row up) and then 3  $\uparrow$  steps.

This is 8 steps in total.

Can the token be moved from r1c9 to r12c4 in fewer than 8 steps?

From above, at least 5  $\swarrow$  steps are needed. These move the token up 5 of the 11 rows necessary.

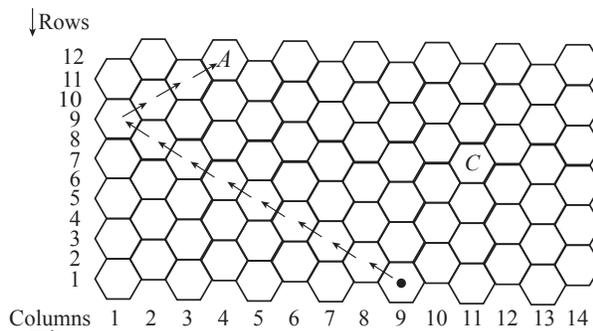
To move the token up the remaining 6 rows, at least 3 more steps are needed, since any step moves the token up at most 2 rows.

Therefore, at least 8 steps are required, so the minimum number of steps required is 8.

(b) We must move the token from r1c9 to r12c4 using as many steps as possible.

Each step moves the token at least 1 row higher, so to move  $12 - 1 = 11$  rows upwards, at most 11 steps can be used.

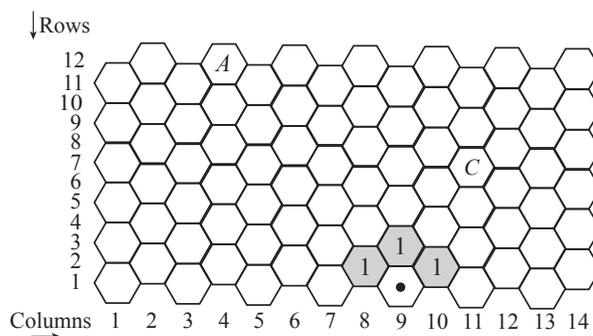
The following diagram shows that this can be done using exactly 11 steps:



Note that no  $\uparrow$  steps are used, since each such step moves the token 2 rows higher. Therefore, the maximum number of steps required is 11. (Note that there are many other 11 step paths which end at A.)

(c) *Solution 1*

After 1 step, there are three different hexagons that can be reached, each in exactly 1 way. (These are the hexagons that can be reached using  $1 \nearrow$ ,  $1 \uparrow$ , or  $1 \nwarrow$ .)



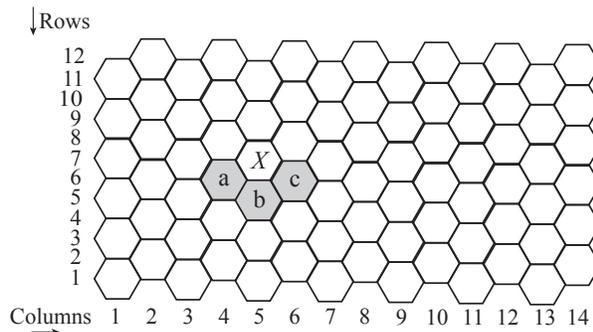
For each positive integer  $s \geq 1$ , we can determine the hexagons that can be reached using exactly  $s + 1$  steps by starting with all of the hexagons that can be reached in exactly  $s$  steps and moving from each of the three possible directions.

Furthermore, we can determine the number of ways that each of these new hexagons can be reached in exactly  $s + 1$  steps.

The number of ways that a particular hexagon can be reached in exactly  $s + 1$  steps is the sum of the number of ways that each of the three hexagons that are adjacent and below can be reached in exactly  $s$  steps.

This is because every path to a particular hexagon that includes exactly  $s + 1$  steps must be made up of a path of exactly  $s$  steps to a hexagon that is adjacent and below followed by a  $\nearrow$  step, a  $\uparrow$  step, or a  $\nwarrow$  step.

For example, in the given diagram, the shaded hexagons can be reached in exactly  $s$  steps in  $a$ ,  $b$  and  $c$  ways, and so the hexagon labelled  $X$  can be reached in exactly  $s + 1$  steps in  $a + b + c$  ways.

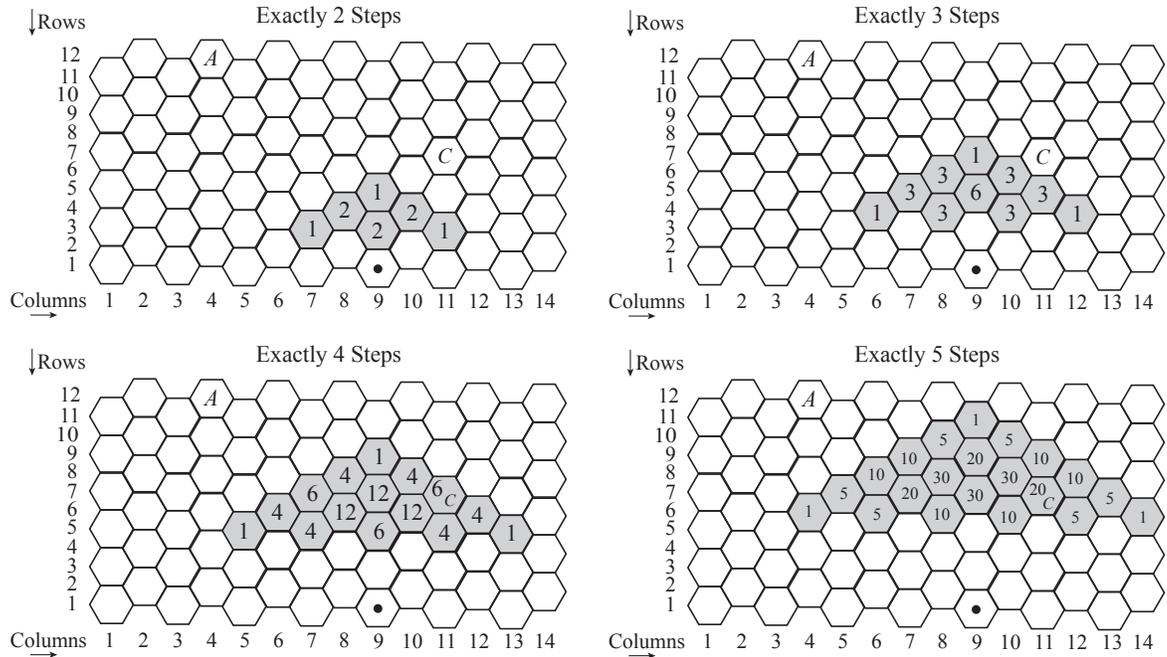


We also note that there are exactly  $3^s$  paths of length  $s$ , since there are 3 possibilities for each of the  $s$  steps in such a path.

This fact allows us to verify at each step that we have accounted for all possibilities.

Using these facts, we carefully determine the hexagons that can be reached in exactly 2, 3, 4, and 5 steps, and the number of ways that each of these hexagons can be reached.

The following diagrams show this information:



From the final diagram, we see that there are exactly 6 hexagons that can be reached in each 5 steps in at least 20 ways. In other words,  $n = 6$ .

*Solution 2*

We note that, when a fixed set of steps from a given path are rearranged, the endpoint of the path does not change.

This is because each step changes the row number and column number in a pre-determined way and so the outcome of each step is not influenced by the other steps in a path.

In other words, the paths  $\nearrow \nwarrow \uparrow \uparrow \nearrow$  and  $\uparrow \nearrow \nearrow \nwarrow \uparrow$  have the same endpoint.

Since there are 3 different kinds of steps ( $\nwarrow, \uparrow, \nearrow$ ), a path with exactly 5 steps can be made up of

- (i) 5 of the same step, or
- (ii) 4 of one step and 1 of another, or
- (iii) 3 of one step and 2 of another, or
- (iv) 3 of one step, 1 of a second type, and 1 of the third type, or
- (v) 2 of one step, 2 of a second type, and 1 of the third type.

These are the only possible combinations of different types of steps.

Each specific collection of 5 steps can possibly be re-arranged in a number of different ways.

We count the number of paths in each category:

- (i) 5 of the same step

There are 3 choices of the type of step.

For each choice, there is only 1 way to arrange the 5 steps.

Therefore, starting from r1c9, there is thus 1 path to each of r6c4 (5 ↖ steps), r11c9 (5 ↑ steps), and r6c14 (5 ↗ steps).

(ii) 4 of one step and 1 of another

There are 3 choices for the type of step to be taken 4 times (we call this choice  $x$ ) and, for each of these choices, there are 2 choices for the type of step to be taken 1 time (we call this choice  $y$ ).

There are thus  $3 \times 2 = 6$  choices for the endpoint of the path (each choice of  $x$  and  $y$  gives a different endpoint and each re-arrangement of a specific set of steps gives the same endpoint).

Given a choice of  $x$  and  $y$ , there are 5 ways to arrange  $xxxxy$  since there are 5 possible positions for the  $y$ , after which the  $x$ 's fill the remaining spots.

Therefore, starting from r1c9, there are thus 5 paths to each of r7c5 ( $x = \swarrow, y = \uparrow$ ), r6c6 ( $x = \swarrow, y = \nearrow$ ), r7c13 ( $x = \nearrow, y = \uparrow$ ), r6c12 ( $x = \nearrow, y = \swarrow$ ), r10c8 ( $x = \uparrow, y = \swarrow$ ), r10c10 ( $x = \uparrow, y = \nearrow$ ).

(iii) 3 of one step and 2 of another

There are 3 choices for the type of step to be taken 3 times (we call this choice  $x$ ) and, for each of these choices, there are 2 choices for the type of step to be taken 2 times (we call this choice  $y$ ).

There are thus  $3 \times 2 = 6$  choices for the endpoint of the path (each choice of  $x$  and  $y$  gives a different endpoint and each re-arrangement of a specific set of steps gives the same endpoint).

Given a choice of  $x$  and  $y$ , there are 10 ways to arrange  $xxxxy$  since there are 10 ways to choose the positions for the two  $y$ 's:

- 1st and 2nd, 1st and 3rd, 1st and 4th, 1st and 5th,
- 2nd and 3rd, 2nd and 4th, 2nd and 5th,
- 3rd and 4th, 3rd and 5th,
- 4th and 5th

after which the  $x$ 's fill the remaining spots.

Therefore, starting from r1c9, there are thus 10 paths to each of r8c6 ( $x = \swarrow, y = \uparrow$ ), r6c8 ( $x = \swarrow, y = \nearrow$ ), r8c12 ( $x = \nearrow, y = \uparrow$ ), r6c10 ( $x = \nearrow, y = \swarrow$ ), r9c7 ( $x = \uparrow, y = \swarrow$ ), r9c11 ( $x = \uparrow, y = \nearrow$ ).

(iv) 3 of one step, 1 of a second type, and 1 of the third type

There are 3 choices for the type of step to be taken 3 times (we call this choice  $x$ ). After  $x$  is chosen, the types of steps to be taken 1 time each are fixed (we call these  $y$  and  $z$  in some order).

There are thus 3 choices for the endpoint of the path (each choice of  $x$  gives a different endpoint and each re-arrangement of a specific set of steps gives the same endpoint).

Given a choice of  $x$ ,  $y$  and  $z$ , there are 20 ways to arrange  $xxxxyz$  since there are 5 ways to choose the position for the  $y$ , after which there are 4 ways to choose the position for the  $z$ , after which the  $x$ 's fill the remaining spots.

Therefore, starting from r1c9, there are thus 20 paths to each of r7c7 ( $x = \swarrow, y = \uparrow, z = \nearrow$ ), r7c11 ( $x = \nearrow, y = \nearrow, z = \uparrow$ ), r9c9 ( $x = \uparrow, y = \swarrow, z = \nearrow$ ).

(v) 2 of one step, 2 of a second type, and 1 of the third type

There are 3 choices for the type of step to be taken 1 time (we call this choice  $z$ ). After  $z$  is chosen, the types of steps to be taken 2 times each are fixed (we call these  $x$  and  $y$ ).

There are thus 3 choices for the endpoint of the path (each choice of  $z$  gives a different endpoint and each re-arrangement of a specific set of steps gives the same endpoint).

Given a choice of  $x$ ,  $y$  and  $z$ , there are 30 ways to arrange  $xyyz$  since there are 10 ways to choose the positions for the  $x$ 's (as there were 10 ways to choose 2 of 5 positions for the  $y$ 's above), after which there are 3 ways to choose the position for the  $z$  (there are three positions left), after which the  $y$ 's fill the remaining spots.

Therefore, starting from r1c9, there are thus 30 paths to each of r8c8 ( $x = \nwarrow$ ,  $y = \uparrow$ ,  $z = \nearrow$ ), r7c9 ( $x = \nearrow$ ,  $y = \nwarrow$ ,  $z = \uparrow$ ), r8c10 ( $x = \uparrow$ ,  $y = \nearrow$ ,  $z = \nwarrow$ ).

Therefore, in summary, there are 6 distinct hexagons (r7c7, r7c11, r9c9, r8c8, r7c9, r8c10) which are the endpoints of at least 20 paths of 5 steps; in other words,  $n = 6$ .



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## ***2014 Fryer Contest***

**Wednesday, April 16, 2014**  
(in North America and South America)

**Thursday, April 17, 2014**  
(outside of North America and South America)

*Solutions*



1. (a) There are 99 positive integers from 1 to 99.  
 The first 9 of these (the integers from 1 to 9) are each single-digit integers and thus contribute  $1 \times 9 = 9$  digits to the given integer.  
 The final 90 positive integers (the integers from 10 to 99) are each 2-digit integers and thus contribute  $2 \times 90 = 180$  digits to the given integer.  
 Therefore, the given integer has  $9 + 180 = 189$  digits in total.
- (b) From part (a), the integer formed by writing the positive integers from 1 to 99 in order next to each other has 189 digits.  
 In addition to these first 99 positive integers, the given integer has the positive integers from 100 to 199 written in order next to one another.  
 Since there are  $199 - 100 + 1 = 100$  of these and each is a 3-digit integer, then they contribute  $3 \times 100 = 300$  more digits.  
 Therefore, the given integer has  $189 + 300 = 489$  digits in total.
- (c) From part (b), the positive integers from 1 to 199 form a 489-digit integer.  
 To obtain 1155 digits, another  $1155 - 489 = 666$  digits are required.  
 Assuming that each of the integers from 200 to  $n$  is a 3-digit integer (we will confirm this), then another  $\frac{666}{3} = 222$  integers are needed to obtain 1155 digits in total.  
 (Since 222 integers beyond the integer 199 are needed to obtain 1155 digits in total, then each of these integers is indeed a 3-digit integer.)  
 Therefore,  $n$  is 222 integers past 199, or  $n = 199 + 222$  and so  $n = 421$ .  
 If the positive integers from 1 to 421 are written in order next to each other, the resulting integer has 1155 digits.
- (d) From part (c), the positive integers from 1 to 421 form a 1155 digit integer.  
 To obtain 1358 digits, another  $1358 - 1155 = 203$  digits are required.  
 Writing the 68 integers that follow 421 will add another  $3 \times 68 = 204$  digits to our total (since each of the 68 integers that follow 421 is a 3-digit integer).  
 The 68<sup>th</sup> integer after 421 is  $421 + 68 = 489$ .  
 However, we only require another 203 digits, not 204. Since the 204<sup>th</sup> digit is the final digit of 489, which is 9, then the 203<sup>rd</sup> digit is 8.  
 Therefore, if the positive integers from 1 to 1000 are written in order next to each other, the 1358<sup>th</sup> digit of the resulting integer is 8.
2. (a) The sum of the 3 angles in any triangle is  $180^\circ$ .  
 Therefore,  $\angle BAC = 180^\circ - \angle ABC - \angle ACB = 180^\circ - 60^\circ - 50^\circ = 70^\circ$ .
- (b) Since  $BD$  bisects  $\angle ABC$ , then  $\angle DBC = \frac{60^\circ}{2} = 30^\circ$ .  
 Since  $CD$  bisects  $\angle ACB$ , then  $\angle DCB = \frac{50^\circ}{2} = 25^\circ$ .  
 Since the sum of the 3 angles in any triangle is  $180^\circ$ ,
- $$\angle BDC = 180^\circ - \angle DBC - \angle DCB = 180^\circ - 30^\circ - 25^\circ = 125^\circ.$$
- (c) In  $\triangle SQR$ , let the measure of  $\angle SQR$  be  $x^\circ$ . Since  $\triangle SQR$  is isosceles with  $QS = RS$ , then  $\angle SRQ = \angle SQR = x^\circ$ .  
 The sum of the 3 angles in  $\triangle SQR$  is  $180^\circ$ , so  $\angle SQR + \angle SRQ + \angle QSR = 180^\circ$ , or  $x^\circ + x^\circ + 140^\circ = 180^\circ$  or  $2x = 40$ , and so  $x = 20$ .  
 Since  $QS$  is the angle bisector of  $\angle PQR$ , then  $\angle PQS = \angle SQR = x^\circ$ .  
 Similarly, since  $RS$  is the angle bisector of  $\angle PRQ$ , then  $\angle PRS = \angle SRQ = x^\circ$ .  
 In  $\triangle PQR$ ,  $\angle PQR = \angle PQS + \angle SQR = (2x)^\circ$  and  $\angle PRQ = \angle PRS + \angle SRQ = (2x)^\circ$ .  
 The sum of the 3 angles in  $\triangle PQR$  is  $180^\circ$ , so  $\angle QPR = 180^\circ - \angle PQR - \angle PRQ$ , or  $\angle QPR = 180^\circ - (2x)^\circ - (2x)^\circ = 180^\circ - (4x)^\circ = 180^\circ - (4 \times 20)^\circ$  or  $\angle QPR = 100^\circ$ .

- (d) We begin by assuming that it is possible that  $\angle QSR = 80^\circ$ .

Proceeding as we did in part (c), if we let the measure of  $\angle SQR$  be  $y^\circ$ , then  $\angle SRQ = \angle SQR = y^\circ$ .

In  $\triangle SQR$ ,  $\angle SQR + \angle SRQ + \angle QSR = 180^\circ$ , or  $y^\circ + y^\circ + 80^\circ = 180^\circ$  or  $2y = 100$ , and so  $y = 50$ .

In  $\triangle PQR$ ,  $\angle PQR = \angle PQS + \angle SQR = (2y)^\circ$  and  $\angle PRQ = \angle PRS + \angle SRQ = (2y)^\circ$  ( $QS$  and  $RS$  are angle bisectors).

Thus,  $\angle QPR = 180^\circ - \angle PQR - \angle PRQ$ , or  $\angle QPR = 180^\circ - (2y)^\circ - (2y)^\circ = 180^\circ - (4y)^\circ$ , and so  $\angle QPR = 180^\circ - (4 \times 50)^\circ$  or  $\angle QPR = -20^\circ$ .

Since all angles must have positive measure, then the only assumption that we made, (that  $\angle QSR = 80^\circ$ ), must be false.

Therefore, it is not possible that  $\angle QSR = 80^\circ$ .

3. (a) Since the base  $BC$  of  $\triangle ABC$  is horizontal, its length is given by the positive difference between the  $x$ -coordinates of points  $B$  and  $C$ , or  $10 - 0 = 10$ .

The height of  $\triangle ABC$  is given by the length of the vertical line segment from point  $A$  to the base  $BC$ , which is the  $y$ -coordinate of point  $A$ , or 9 (since  $BC$  lies along the  $x$ -axis). Thus, the area of  $\triangle ABC$  before the first move in the game is made is  $\frac{1}{2} \times 10 \times 9 = 45$ .

- (b) At this point in the game, the area of  $\triangle ABC$  is  $\frac{1}{2} \times 10 \times 7$  or 35 (since the length of the base is still 10 and the height is 7).

The person who makes the area of  $\triangle ABC$  equal to 25 wins the game.

The base of  $\triangle ABC$ ,  $BC$ , remains fixed at length 10 throughout the game since only point  $A$  can move.

The area of  $\triangle ABC$  is 25 when the height of the triangle is 5, since  $\frac{1}{2} \times 10 \times 5 = 25$ .

The height of  $\triangle ABC$  is 5 when the  $y$ -coordinate of point  $A$  is 5.

That is, the only way for a player to win the game is to make the move that changes the  $y$ -coordinate of point  $A$  to 5.

It is now Dexter's turn to move. Dexter can move  $A$  down one unit to the point  $(2, 6)$  or he may move  $A$  left one unit to the point  $(1, 7)$ .

If Dexter moves  $A$  down one unit to the point  $(2, 6)$ , then on the next turn Ella can move  $A$  down one more unit to the point  $(2, 5)$  and win the game (since the  $y$ -coordinate would be 5).

Alternatively, if Dexter moves  $A$  one unit left to the point  $(1, 7)$ , then on the next turn Ella can move  $A$  one more unit left to the point  $(0, 7)$  (Ella would not choose to move  $A$  down one unit to  $(1, 6)$  since then Dexter could win on his next move).

On the next turn, Dexter is forced to move  $A$  down one unit to  $(0, 6)$  since moving  $A$  left would make the  $x$ -coordinate negative which is not permitted.

On the next move, Ella can move  $A$  down one more unit to  $(0, 5)$  and win the game.

We have shown that for all possible moves that Dexter can make, Ella can always win the game from the point  $(2, 7)$ .

- (c) (i) Beginning with point  $A$  at  $(6, 9)$ , the player who moves second, Geoff, has the winning strategy.

This winning strategy will be first described and then justified in parts (ii) and (iii) below.

- (ii) There are many different ways to describe Geoff's winning strategy.

One way to describe it is to say that on Geoff's turn, he will perform the same move (that is, move  $A$  either left or down) that his opponent Faisal just performed in the previous move.

- (iii) Why does Geoff win the game every time if he always matches the move that Faisal just performed?

Begin by assuming that Geoff can always match Faisal's move (an assumption that we will prove later on).

At the start of the game, the  $y$ -coordinate of point  $A$  is 9.

If Faisal moves down to 8, then Geoff will match and move down to 7.

If Faisal moves down to 6, then Geoff will match and move down to 5.

From part (b), we know that the player who makes the  $y$ -coordinate of point  $A$  equal to 5 wins the game.

Thus Geoff wins the game, assuming that Geoff can always move down after Faisal and is never forced to move down before Faisal.

Next, we must prove our assumption that Geoff can always match Faisal's move.

First, consider moves down.

A move down followed by a matching move down is always possible since the game will end when the  $y$ -coordinate of  $A$  is 5 (and as was already shown, it is Geoff who makes this final move).

That is, the  $y$ -coordinate of  $A$  will never be made negative and Geoff can always match any down move that Faisal makes.

Finally, we consider moves left.

At the start of the game, the  $x$ -coordinate of point  $A$  is 6, an even integer.

If at any time in the game Faisal moves  $A$  left one unit, then the  $x$ -coordinate becomes 5, an odd integer.

On his next turn, Geoff matches Faisal's move (left), and the  $x$ -coordinate becomes 4, an even integer again.

That is, each time Faisal moves  $A$  to the left, the  $x$ -coordinate becomes an odd integer, and since Geoff matches Faisal's move he returns the  $x$ -coordinate to an even integer.

Since the smallest  $x$ -coordinate that is permitted is an even integer, 0 (coordinates cannot be made negative), it is Geoff who will (if required) make the last possible move to the left.

On his next turn, Faisal is then forced to move  $A$  downward.

Therefore we have shown that Geoff will always be able to match Faisal's move, and that as a result, he will be the player who moves point  $A$  so that the  $y$ -coordinate becomes 5, thus winning the game.

4. (a) The eight subsets of the set  $\{1, 2, 3\}$  are:  $\{\}$ ,  $\{1\}$ ,  $\{2\}$ ,  $\{3\}$ ,  $\{1, 2\}$ ,  $\{1, 3\}$ ,  $\{2, 3\}$ , and  $\{1, 2, 3\}$ . Therefore the eight subset sums are: 0, 1, 2, 3, 3, 4, 5, and 6.

- (b) When constructing subsets, any element of the original set may either be included in the subset or not included in the subset.

That is, when constructing a subset from the set  $\{1, 2, 3, 4, 5\}$ , there are two choices for each of the 5 elements of the set; include it in the subset or do not.

Since there are two choices for each of the 5 elements, there are a total of  $2^5$  or 32 different subsets that can be created.

Consider the number of these 32 subsets which contain the number 1.

That is, when counting the number of these subsets, there is only one choice for the number one (include it in the subset), while there are still two choices for each of the numbers 2, 3, 4, and 5.

Thus, there are  $2^4$  or 16 subsets that contain the number 1 (and so there are  $32 - 16 = 16$  subsets which do not contain the number 1).

This is true in general for each of the elements of the original set; the number 2 will be included in 16 of the 32 subsets, the number 3 will be included in 16 subsets, and likewise for the numbers 4 and 5.

Since the element 1 is contained in exactly 16 subsets, then these 16 occurrences of 1 contribute exactly  $1 \times 16 = 16$  to the sum of the subsets that they appear in.

Similarly, since the element 2 is contained in exactly 16 subsets, then these 16 occurrences of 2 contribute exactly  $2 \times 16 = 32$  to the sum of the subsets that they appear in.

Since each of the 5 elements is contained in exactly 16 subsets, then the total contribution of all 5 elements in all possible subsets is

$$(1 \times 16) + (2 \times 16) + (3 \times 16) + (4 \times 16) + (5 \times 16) = 16(1 + 2 + 3 + 4 + 5) = 16(15) = 240.$$

Thus, the sum of all of the subset sums of  $\{1, 2, 3, 4, 5\}$  is 240.

- (c) Each subset of  $\{1, 3, 4, 5, 7, 8, 12, 16\}$  consists of zero or more of the odd integers  $\{1, 3, 5, 7\}$  and zero or more of the multiples of four  $\{4, 8, 12, 16\}$ .

If a subset is formed using only numbers that are divisible by 4, then the subset sum must also be divisible by 4 (since the sum of multiples of 4 is a multiple of 4).

So then every subset of  $\{4, 8, 12, 16\}$  must have a subset sum that is divisible by 4.

As in part (b), each of the 4 elements will appear in  $2^3 = 8$  of the  $2^4 = 16$  subsets. The sum of all subset sums using only elements from the set  $\{4, 8, 12, 16\}$  is then

$$(4 \times 8) + (8 \times 8) + (12 \times 8) + (16 \times 8) = 8(4 + 8 + 12 + 16) = 320.$$

Any subset includes 0, 1, 2, 3 or 4 of the elements of  $\{4, 8, 12, 16\}$ .

The sum of these elements is divisible by 4.

For the sum of all elements in the subset to be divisible by 4, the sum of the remaining (odd) elements must be divisible by 4.

Some combinations of the remaining (odd) elements  $\{1, 3, 5, 7\}$  produce sums which are divisible by 4.

Specifically, these are  $\{1, 3\}$ ,  $\{1, 7\}$ ,  $\{3, 5\}$ ,  $\{5, 7\}$ ,  $\{1, 3, 5, 7\}$ , and the empty set  $\{\}$ .

(No other subsets of  $\{1, 3, 5, 7\}$  have a sum divisible by 4 since each of the individual elements is odd, thus the sum of any 3 of them is odd, and the only pairs remaining are  $\{1, 5\}$  and  $\{3, 7\}$ , whose sums are 6 and 10).

Since each of these subsets has a sum which is divisible by 4, then combining them (that is, using all of their elements) with any of the 16 subsets of  $\{4, 8, 12, 16\}$  (including the empty set) will produce a new subset whose sum is also divisible by 4.

What remains is to determine the sum of all of these subset sums.

First consider combining both of the elements  $\{1, 3\}$  with each of the 16 subsets of  $\{4, 8, 12, 16\}$ . As we previously determined, the sum of all of the subsets of  $\{4, 8, 12, 16\}$  is 320.

Combining both of the elements  $\{1, 3\}$  with each of these subsets adds  $1 + 3 = 4$  to each of the subset sums.

Since there are 16 subsets, then the sum of all subset sums of this form is  $(4 \times 16) + 320$  or 384.

Similarly, combining both of the elements  $\{1, 7\}$  with each of the subsets of  $\{4, 8, 12, 16\}$  adds  $1 + 7 = 8$  to each of the subset sums.

Since there are 16 subsets, then the sum of all subset sums of this form is  $(8 \times 16) + 320$  or 448.

The sum of all subset sums which combine both  $\{3, 5\}$  with all subsets of  $\{4, 8, 12, 16\}$  is  $(8 \times 16) + 320$  or 448.

The sum of all subset sums which combine both  $\{5, 7\}$  with all subsets of  $\{4, 8, 12, 16\}$  is

$(12 \times 16) + 320$  or 512.

The sum of all subset sums which combine all of  $\{1, 3, 5, 7\}$  with all subsets of  $\{4, 8, 12, 16\}$  is  $(16 \times 16) + 320$  or 576.

Finally, combining the empty set with each of the subsets of  $\{4, 8, 12, 16\}$  adds nothing to each of the subset sums and so the sum of all subset sums is 320 ( $(0 \times 16) + 320 = 320$ ).

Since there are no other combinations of elements which produce subset sums that are divisible by 4, the sum of all required subset sums is  $384+448+448+512+576+320 = 2688$ .



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## ***2013 Fryer Contest***

**Thursday, April 18, 2013**  
(in North America and South America)

**Friday, April 19, 2013**  
(outside of North America and South America)

*Solutions*

1. (a) Ann's average score for these two games was  $\frac{103+117}{2} = \frac{220}{2}$  or 110.

(b) *Solution 1*

Since the average score of Bill's three games was 115, then the sum of his three scores was  $115 \times 3 = 345$ .

The sum of Bill's scores in his first two games was  $108 + 125 = 233$ .

Therefore, Bill's score in his third game was  $345 - 233 = 112$ .

*Solution 2*

Let  $x$  represent Bill's score in the third game.

Since the average of the three games was 115, then  $\frac{108 + 125 + x}{3} = 115$ .

Solving for  $x$ , we get  $108 + 125 + x = 115 \times 3$  or  $233 + x = 345$ , so  $x = 345 - 233 = 112$ .

Therefore, Bill's score in his third game was 112.

(c) *Solution 1*

Since the average score of Cathy's first three games was 113, then the sum of her first three scores was  $113 \times 3 = 339$ .

If the average score of Cathy's first five games was 120, then the sum of her five scores was  $120 \times 5 = 600$ .

Therefore, the sum of Cathy's scores in games four and five was  $600 - 339 = 261$ .

Since Cathy scored the same in her fourth and fifth games, then each of these scores would equal  $\frac{261}{2} = 130.5$ .

However, in bowling, each score is a whole number so this is not possible.

Therefore, it is not possible for Cathy to have an average score of 120 in these five games.

*Solution 2*

Since the average score of Cathy's first three games was 113, then the sum of her first three scores was  $113 \times 3 = 339$ .

Let  $y$  represent Cathy's score in game four and her score in game five (since they were the same score).

Since the average of the five games was 120, then  $\frac{339 + y + y}{5} = 120$ .

Solving for  $y$ , we get  $339 + 2y = 120 \times 5$  or  $339 + 2y = 600$ , so  $2y = 261$  and  $y = 130.5$ .

However, in bowling, each score is a whole number so this is not possible.

Therefore, it is not possible for Cathy to have an average score of 120 in these five games.

2. (a) The outside of the field consists of two straight sides each of length 100 m joined by two semi-circular arcs each of diameter 60 m.

Thus, the perimeter of the field is equal to the total length of the two straight sides, or 200 m, added to the circumference of a circle with diameter 60 m, or  $\pi(60)$  m.

Thus, the perimeter of the field is  $(200 + 60\pi)$  m.

(b) Amy may run along the perimeter of the field from  $C$  to  $D$  in two possible ways.

No matter which way Amy chooses, she will travel along one of the two straight sides, and one of the two semi-circular arcs.

That is, Amy will travel exactly one half of the total perimeter of the field, or  $(100+30\pi)$  m, in travelling from  $C$  to  $D$ .

Billais runs in a straight line distance from  $C$  to  $D$ .

Since each straight 100 m side is perpendicular to each 60 m diameter, then  $CD$  is the hypotenuse of a right triangle with the other two sides having length 100 m and 60 m.

By the Pythagorean Theorem,  $CD^2 = 100^2 + 60^2$  or  $CD = \sqrt{13600}$  m (since  $CD > 0$ ).  
Thus, Amy travels  $100 + 30\pi - \sqrt{13600} \approx 78$  m farther than Billais.

- (c) The outside of the track consists of two straight sides each of length 100 m joined by two semi-circular arcs each of diameter  $60 + x + x$  or  $(60 + 2x)$  m.

Thus, the perimeter of the outside of the track is equal to the total length of the two straight sides, or 200 m, added to the circumference of a circle with diameter  $(60 + 2x)$  m, or  $\pi(60 + 2x)$  m.

Thus, the perimeter of the outside of the track is  $(200 + (60 + 2x)\pi)$  m.

Since the perimeter of the outside of the track is 450 m, then  $200 + (60 + 2x)\pi = 450$ .

Solving this equation for  $x$ , we get  $(60 + 2x)\pi = 250$  or  $60 + 2x = \frac{250}{\pi}$  or  $2x = \frac{250}{\pi} - 60$  or  $x = \frac{1}{2}(\frac{250}{\pi} - 60)$ , and so  $x = \frac{125}{\pi} - 30 \approx 9.789$ .

Rounded to the nearest whole number, the value of  $x$  is 10.

3. (a) The four-digit number  $51A3$  is divisible by 3 when the sum of its digits is divisible by 3.

The sum of its digits is  $5 + 1 + A + 3 = 9 + A$ .

The values for the digit  $A$  such that  $9 + A$  is divisible by 3 are 0, 3, 6, or 9.

- (b) The four-digit number  $742B$  is divisible by 2 when it is even.

Thus,  $742B$  is divisible by 2 if  $B$  is 0, 2, 4, 6, or 8.

The four-digit number  $742B$  must also be divisible by 3.

The number  $742B$  is divisible by 3 when  $7 + 4 + 2 + B$  or  $13 + B$  is divisible by 3.

Thus,  $742B$  is divisible by 3 if  $B$  is 2, 5 or 8.

The only values of the digit  $B$  such that the four-digit number  $742B$  is divisible by 6 (divisible by both 2 and 3), are  $B = 2$  or  $B = 8$ .

- (c) An integer is divisible by 15 when it is divisible by both 5 and 3 (since the product of 5 and 3 is 15 and since 5 and 3 have no common factor).

The integer  $1234PQPQ$  is divisible by 5 when its units digit,  $Q$ , is 0 or 5.

The integer  $1234PQPQ$  is divisible by 3 when the sum of its digits,  $1 + 2 + 3 + 4 + P + Q + P + Q$  or  $10 + 2P + 2Q$ , is divisible by 3.

We proceed by checking the two cases,  $Q = 0$  and  $Q = 5$ .

When  $Q = 0$ , the sum of the digits  $10 + 2P + 2Q$  is  $10 + 2P$ .

Therefore, when  $Q = 0$  the values of  $P$  for which  $10 + 2P$  is divisible by 3 are 1, 4 and 7.

When  $Q = 5$ , the sum of the digits  $10 + 2P + 2Q$  is  $10 + 2P + 10$  or  $20 + 2P$ .

Therefore when  $Q = 5$ , the values of  $P$  for which  $20 + 2P$  is divisible by 3 are 2, 5 and 8.

The values of digits  $P$  and  $Q$  (written as  $(P, Q)$ ) for which  $1234PQPQ$  is divisible by 15 are  $(1, 0)$ ,  $(4, 0)$ ,  $(7, 0)$ ,  $(2, 5)$ ,  $(5, 5)$ , or  $(8, 5)$ .

- (d) An integer is divisible by 12 when it is divisible by both 4 and 3 (since the product of 4 and 3 is 12 and since 4 and 3 have no common factor).

In the product  $2CC \times 3D5$ ,  $3D5$  is odd and thus cannot be divisible by 4.

Therefore,  $2CC$  must be divisible by 4 and so must also be divisible by 2.

That is,  $C$  must be an even number. We proceed by checking  $C = 0, 2, 4, 6, 8$ .

If  $C = 0$ , 200 is divisible by 4 and so  $C = 0$  is a possibility.

If  $C = 2$ , 222 is not divisible by 4 and so  $C = 2$  is not a possibility.

If  $C = 4$ , 244 is divisible by 4 and so  $C = 4$  is a possibility.

If  $C = 6$ , 266 is not divisible by 4 and so  $C = 6$  is not a possibility.

Finally, if  $C = 8$ , 288 is divisible by 4 and so  $C = 8$  is a possibility.

We now proceed by using these three cases,  $C = 0$ ,  $C = 4$  and  $C = 8$ , to determine possible values for  $D$ .

When  $C = 0$ , the product  $2CC \times 3D5$  becomes  $200 \times 3D5$ .

The number 200 is not divisible by 3 since the sum of its digits, 2, is not divisible by 3. Therefore,  $3D5$  must be divisible by 3 and so  $8 + D$  is divisible by 3.

The possible values for  $D$  such that  $8 + D$  is divisible by 3 are 1, 4 and 7.

In this case, there are 3 possible pairs of digits  $C$  and  $D$  for which the product  $2CC \times 3D5$  is divisible by 12.

When  $C = 4$ , the product  $2CC \times 3D5$  becomes  $244 \times 3D5$ .

The number 244 is not divisible by 3 since the sum of its digits, 10, is not divisible by 3. Therefore,  $3D5$  must be divisible by 3 and as was shown in the first case, there are 3 possible pairs of digits  $C$  and  $D$  for which the product  $2CC \times 3D5$  is divisible by 12.

When  $C = 8$ , the product  $2CC \times 3D5$  becomes  $288 \times 3D5$ .

The number 288 is divisible by 3 since the sum of its digits, 18, is divisible by 3.

Therefore, the product  $288 \times 3D5$  is divisible by 3 independent of the value of  $D$ .

That is,  $D$  can be any digit from 0 to 9.

In this case, there are 10 possible pairs of digits  $C$  and  $D$  for which the product  $2CC \times 3D5$  is divisible by 12.

The total number of pairs of digits  $C$  and  $D$  such that  $2CC \times 3D5$  is divisible by 12 is  $3 + 3 + 10 = 16$ .

4. (a) Since the dot starts at  $(0, 0)$  and ends at  $(1, 0)$ , it moves a total of 1 unit right and 0 units up or down.

Since the dot moves 0 units up or down, then the number of  $\uparrow$  moves equals the number of  $\downarrow$  moves.

Since the total number of moves is at most 4, then the moves include 0  $\uparrow$  and 0  $\downarrow$  (0 moves in total), or 1  $\uparrow$  and 1  $\downarrow$  (2 moves in total), or 2  $\uparrow$  and 2  $\downarrow$  (4 moves in total).

Since the dot moves 1 unit right, then the number of  $\rightarrow$  moves is 1 more than the number of  $\leftarrow$  moves.

Since the total number of moves is at most 4, then there are 1  $\rightarrow$  and 0  $\leftarrow$  (1 move in total), or 2  $\rightarrow$  and 1  $\leftarrow$  (3 moves in total).

If there are 0 up/down moves, there can be 1 or 3 left/right moves.

If there are 2 up/down moves, there can be 1 left/right move.

If there are 4 up/down moves, there can be 0 left/right moves, which isn't possible.

With 0 up/down and 1 left/right moves, the sequence of moves is  $\rightarrow$ .

With 0 up/down and 3 left/right moves, the sequence of moves is  $\leftarrow\rightarrow\rightarrow$  or any arrangement of these moves.

With 2 up/down and 1 left/right moves, the sequence of moves is  $\uparrow\downarrow\rightarrow$  or any arrangement of these moves.

There are 3 possible arrangements of the moves  $\leftarrow\rightarrow\rightarrow$ . (This is because the  $\leftarrow$  can be the first, second or third moves, and the two remaining moves are both  $\rightarrow$ .)

There are 6 possible arrangements of the moves  $\uparrow\downarrow\rightarrow$ . (This is because there are three moves that can go first, and then two remaining moves that can go second, and then one move that can go third, and so  $3 \times 2 \times 1 = 6$  combinations of moves.)

Therefore, there are  $1 + 3 + 6 = 10$  possible ways for the dot to end at  $(1, 0)$ .

- (b) If the dot makes exactly 4 moves, these moves can consist of

- any combination of 4 right/left moves, or
- any combination of 3 right/left moves and any 1 up/down move, or
- any combination of 2 right/left moves and any combination of 2 up/down moves, or

- any 1 right/left move and any combination of 3 up/down moves, or
- any combination of 4 up/down moves.

Given a specific set of moves, any arrangement of these moves will move the dot to the same point, so we can think, for example, of all of the right/left moves occurring first followed by the up/down moves.

Four right/left moves can be 4 right (ending at  $x = 4$ ), 3 right and 1 left (ending at  $x = 3 - 1 = 2$ ), 2 right and 2 left (ending at  $x = 0$ ), 1 right and 3 left (ending at  $x = 1 - 3 = -2$ ), or 4 left (ending at  $x = -4$ ).

Three right/left moves can be 3 right, 2 right and 1 left, 1 right and 2 left, or 3 left, ending at  $x = 3, 1, -1, -3$ , respectively.

Two right/left moves can be 2 right, 1 right and 1 left, or 2 left, ending at  $x = 2, 0, -2$ , respectively.

One right/left move can be 1 right or 1 left, ending at  $x = 1, -1$ , respectively.

No moves right or left will end at  $x = 0$ .

Similarly, four up/down moves can end at  $y = 4, 2, 0, -2, -4$ , three up/down moves can end at  $y = 3, 1, -1, -3$ , two up/down moves can end at  $y = 2, 0, -2$ , one up/down move can end at  $y = 1, -1$ , and no moves up or down will end at  $y = 0$ . (To see this, we can repeat the previous argument, replacing “right/left” with “up/down” and changing  $x$ -coordinates to  $y$ -coordinates.)

Given the total number of moves in each of the horizontal and vertical direction, any combination of the resulting  $x$ - and  $y$ -coordinates is possible, since the horizontal and vertical moves do not affect each other.

Therefore, if the dot makes exactly 4 moves, we have

- any combination of 4 right/left moves and no moves up or down ( $x = 4, 2, 0, -2, -4$ , and  $y = 0$ ), ending at the points  $(4, 0), (2, 0), (0, 0), (-2, 0), (-4, 0)$ , or
- any combination of 3 right/left moves and any 1 up/down move ( $x = 3, 1, -1, -3$ , and  $y = 1, -1$ ), ending at the points  $(3, 1), (3, -1), (1, 1), (1, -1), (-1, 1), (-1, -1), (-3, 1), (-3, -1)$ , or
- any combination of 2 right/left moves and any combination of 2 up/down moves ( $x = 2, 0, -2$ , and  $y = 2, 0, -2$ ), ending at the points  $(2, 2), (2, 0), (2, -2), (0, 2), (0, 0), (0, -2), (-2, 2), (-2, 0), (-2, -2)$ , or
- any 1 right/left move and any combination of 3 up/down moves ( $x = 1, -1$ , and  $y = 3, 1, -1, -3$ ), ending at the points  $(1, 3), (1, 1), (1, -1), (1, -3), (-1, 3), (-1, 1), (-1, -1), (-1, -3)$ , or
- any combination of 4 up/down moves and no moves right or left ( $x = 0$ , and  $y = 4, 2, 0, -2, -4$ ), ending at the points  $(0, 4), (0, 2), (0, 0), (0, -2), (0, -4)$ .

After careful observation of the list of ending points above, we see that a number of points can be reached in more than one way.

In counting the total number of distinct points we must take care to count the same point more than once.

In the first bullet above, there are 5 distinct points.

In the second bullet, there are 8 new points that have not yet been counted.

In the third bullet there are 9 points, however, we have already counted the 3 points  $(2, 0), (0, 0)$ , and  $(-2, 0)$ .

In the fourth bullet there are 8 points, however, we have already counted the 4 points  $(1, 1), (1, -1), (-1, 1)$ , and  $(-1, -1)$ .

In the final bullet there are 5 points, however, we have already counted the 3 points

$(0, 2)$ ,  $(0, 0)$ , and  $(0, -2)$ .

Therefore, the total number of possible points is  $5 + 8 + (9 - 3) + (8 - 4) + (5 - 3) = 25$ .

- (c) The dot can get to  $(-7, 12)$  in 19 moves: 7 left and 12 up.

We need at least 19 moves for the dot to get to  $(-7, 12)$ , since we need at least 7 moves to the left and at least 12 moves up (with possibly more moves that cancel each other out).

The dot can also get to  $(-7, 12)$  in 21 moves: 7 left, 12 up, 1 left, 1 right.

The dot can also get to  $(-7, 12)$  in 23 moves: 7 left, 12 up, 2 left, 2 right.

In a similar way, the dot can get to  $(-7, 12)$  in  $19 + 2m$  moves for any non-negative integer  $m$ : 7 left, 12 up,  $m$  left,  $m$  right.

As  $m$  ranges from 0 to 40, the expression  $19 + 2m$  takes every odd integer value from 19 to 99.

This is 41 integer values.

We have shown that if  $k < 19$ , then the dot cannot get to  $(-7, 12)$  in  $k$  moves, and if  $k \geq 19$  and  $k$  is odd, then the dot can get to  $(-7, 12)$  in  $k$  moves.

Finally, we show that if  $k \geq 19$  and  $k$  is even, then the dot cannot get to  $(-7, 12)$  in  $k$  moves. This will complete our solution, and give an answer of 41, as above.

Suppose that  $k \geq 19$  is even and that the dot can get to  $(-7, 12)$  in  $k$  moves.

These  $k$  moves include  $h$  right/left moves and  $v$  up/down moves, with  $k = h + v$ .

Since  $k$  is even, then  $h$  and  $v$  are both even or both odd. (This is because Even + Even is Even, Odd + Odd is Even, and Odd + Even is Odd.)

Suppose that  $h$  is even. The  $h$  right/left moves include  $r$  right moves and  $l$  left moves, which gives  $h = r + l$  and  $r - l = -7$ , since the dot ends with  $x$ -coordinate  $-7$ .

Since  $h$  is even and  $h = r + l$ , then  $r$  and  $l$  are both even or both odd.

But if  $r$  and  $l$  are both even or both odd, then  $r - l$  must be even.

This disagrees with the fact that  $r - l = -7$ .

Therefore,  $h$  cannot be even, and so  $h$  is odd.

Since  $h$  is odd, then  $v$  is odd.

But the  $v$  up/down moves include  $u$  up moves and  $d$  down moves, which gives  $v = u + d$  and  $u - d = 12$ .

Since  $v = u + d$  is odd, then  $u$  and  $d$  are odd and even or even and odd, which means that  $u - d$  is odd, and so cannot be equal to 12.

Therefore, none of the possibilities work.

This means that we cannot get to  $(-7, 12)$  in an even number of moves.

Thus, there are 41 positive integers  $k$  with  $k \leq 100$  for which the dot can reach  $(-7, 12)$  in  $k$  moves.

- (d) Consider any point  $P$  at which the dot can end in 47 moves. The dot can also end at this point in 49 moves by adding, for example, a cancelling pair such as  $\rightarrow\leftarrow$  onto the end of the sequence of 47 moves ending at  $P$ .

Therefore, each of the 2304 points at which the dot can end in 47 moves is a point at which the dot can end in 49 moves.

Any other point  $Q$  at which the dot can end in 49 moves cannot be arrived at in 47 moves. Consider any sequence of moves that takes the dot to  $Q$ . It cannot include both a right and a left move, or both an up and a down move, otherwise these moves could be "cancelled" and removed, giving a sequence of 47 moves ending at  $Q$ .

Thus, a sequence of moves ending at  $Q$  includes  $h$  horizontal moves and  $v$  vertical moves (with  $h + v = 49$ ), where the horizontal moves are either all right or all left moves, and the vertical moves are either all up or all down moves.

If  $h = 0$ , then  $v = 49$ , so the dot can end at  $(0, 49)$  (49 moves up) or at  $(0, -49)$  (49 moves

down).

Similarly, if  $v = 0$ , the dot can end at  $(49, 0)$  or  $(-49, 0)$ .

If  $h$  and  $v$  are both not zero, then there are both horizontal and vertical moves.

We consider the case where there are  $h$  right moves and  $v$  up moves, with  $h + v = 49$ .

Since  $h \geq 1$  and  $v \geq 1$ , then  $h$  can take any value from 1 to 48 giving values of  $v$  from 48 to 1, and points  $(1, 48), (2, 47), \dots, (47, 2), (48, 1)$ .

This is exactly 48 distinct points.

Similarly, in each of the cases of right moves and down moves, left moves and up moves, and left moves and down moves, we obtain 48 distinct points.

Therefore, the number of points  $Q$  at which the dot can end in 49, but not 47 moves, is  $2 + 2 + 4 \times 48 = 196$ .

This gives a total of  $2304 + 196 = 2500$  points at which the dot can end in exactly 49 moves.



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## *2012 Fryer Contest*

Thursday, April 12, 2012  
(in North America and South America)

Friday, April 13, 2012  
(outside of North America and South America)

*Solutions*

1. (a) Candidate A received  $\frac{1008}{5600} \times 100\% = 0.18 \times 100\%$  or 18% of all votes.

(b) *Solution 1*

Since  $\frac{3}{5} \times 100\% = 0.60 \times 100\%$ , Candidate B received 60% of all votes.

Since Candidates C and D tied, they equally shared the remaining  $100\% - 60\% = 40\%$  of the votes.

Therefore, Candidate C received  $\frac{1}{2}$  of 40% of the votes, or 20% of all votes.

*Solution 2*

Since Candidate B received  $\frac{3}{5}$  of all votes, then Candidates C and D shared the remaining  $1 - \frac{3}{5} = \frac{2}{5}$  of all votes.

Candidates C and D tied, thus they shared equally the remaining  $\frac{2}{5}$  of the votes.

Therefore, Candidate C received  $\frac{1}{2}$  of  $\frac{2}{5}$  or  $\frac{1}{2} \times \frac{2}{5} = \frac{2}{10} = \frac{1}{5}$  of the votes.

Since  $\frac{1}{5} \times 100\% = 0.20 \times 100\%$ , Candidate C received 20% of all votes.

(c) *Solution 1*

At 10:00 p.m., 90% of 6000 votes or  $\frac{90}{100} \times 6000 = 5400$  votes had been counted.

Of those 5400 votes that had been counted, Candidate E received 53%.

Therefore at 10 p.m.,  $\frac{53}{100} \times 5400 = 2862$  votes had been counted for Candidate E.

Since there were only 2 candidates, the remaining  $5400 - 2862$  or 2538 votes must have been counted for Candidate F.

Thus, there were  $2862 - 2538$  or 324 more votes counted for Candidate E than for Candidate F.

*Solution 2*

At 10:00 p.m., 90% of 6000 votes or  $\frac{90}{100} \times 6000 = 5400$  votes had been counted.

Of those 5400 votes that had been counted, Candidate E received 53%.

Since there are only 2 candidates, then Candidate F must have received the remaining  $100\% - 53\%$  or 47%.

Thus, Candidate E received  $53\% - 47\%$  or 6% more votes than Candidate F.

Since there were a total of 5400 votes that had been counted at 10:00p.m., then Candidate E received 6% of 5400 or 324 more votes than Candidate F.

(d) Candidate H received 40% of the votes and Candidate J received 35% of the votes.

Thus, the only other candidate, G, received the remaining  $100\% - 40\% - 35\% = 25\%$  of the votes.

Since Candidate G received 2000 votes representing 25% of all votes cast, then the total number of votes cast was  $2000 \times 4 = 8000$  (since  $25\% \times 4 = 100\%$ ).

Thus, Candidate H received 40% of 8000 votes or  $\frac{40}{100} \times 8000 = 3200$  votes.

2. (a) Factoring gives  $112 = 2 \times 56 = 2 \times 2 \times 28 = 2 \times 2 \times 2 \times 14 = 2 \times 2 \times 2 \times 2 \times 7$ .

So, the prime factorization of 112 is  $2 \times 2 \times 2 \times 2 \times 7$  or  $2^4 \times 7$ .

(b) For every perfect square, each of its prime factors occurs an even number of times (see the *Note* at the end of part (d) for a brief explanation of this).

From part (a), the prime factorization of 112 is  $2^4 \times 7$ .

We are asked for the smallest value of the positive integer  $u$  so that  $112 \times u$  or  $2^4 \times 7 \times u$  is a perfect square.

The prime factor 2 already occurs an even number of times, (four times), in the factorization of 112.

Thus, no additional factors of 2 are needed to make  $2^4 \times 7 \times u$  a perfect square.

However, the prime factor 7 occurs only once.

Since all prime factors must occur an even number of times, then at least one additional

factor of 7 is needed for  $2^4 \times 7 \times u$  to be a perfect square.

Therefore, the smallest positive integer  $u$  that makes the product  $112 \times u$  a perfect square, is 7:

$$112 \times u = 2^4 \times 7 \times u = 2^4 \times 7 \times 7 = 2^4 \times 7^2 = (2^2 \times 7) \times (2^2 \times 7).$$

- (c) Since  $5632 = 512 \times 11$  and  $512 = 2^9$ , then the prime factorization of 5632 is  $2^9 \times 11$ .

Again, every perfect square has each of its prime factors occurring an even number of times. We are asked for the smallest value of the positive integer  $v$  so that  $5632 \times v$  or  $2^9 \times 11 \times v$  is a perfect square.

The prime factor 2 occurs an odd number of times, (nine times), in the factorization of 5632.

Thus, at least one additional factor of 2 is needed to make  $2^9 \times 11 \times v$  a perfect square.

The prime factor 11 occurs only once.

Thus, at least one additional factor of 11 is needed for  $2^9 \times 11 \times v$  to be a perfect square.

Therefore, the smallest positive integer  $v$  that makes the product  $5632 \times v$  a perfect square, is  $2 \times 11$  or 22:

$$5632 \times v = 2^9 \times 11 \times v = 2^9 \times 11 \times 2 \times 11 = 2^{10} \times 11^2 = (2^5 \times 11) \times (2^5 \times 11).$$

- (d) For every perfect cube, the number of times that each of its prime factors occurs is a multiple of 3 (see the *Note* below for a brief explanation of this).

From part (a), the prime factorization of 112 is  $2^4 \times 7$ .

We are asked for the smallest value of the positive integer  $w$  so that  $112 \times w$  or  $2^4 \times 7 \times w$  is a perfect cube.

The prime factor 2 occurs four times in the factorization of 112.

Thus, the smallest number of additional factors of 2 needed to make  $2^4 \times 7 \times w$  a perfect cube is two (since 6 is the smallest multiple of 3 that is greater than 4).

The prime factor 7 occurs only once.

Thus, the smallest number of additional factors of 7 needed to make  $2^4 \times 7 \times w$  a perfect cube is two (since 3 is the smallest multiple of 3 that is greater than 1).

Therefore, the smallest positive integer  $w$  that makes the product  $112 \times w$  a perfect cube, is  $2^2 \times 7^2$  or 196:

$$112 \times w = 2^4 \times 7 \times w = 2^4 \times 7 \times 2^2 \times 7^2 = 2^6 \times 7^3 = (2^2 \times 7) \times (2^2 \times 7) \times (2^2 \times 7).$$

*Note:* Every positive integer greater than 1 can be written as a unique product of prime numbers (this is known as the Fundamental Theorem of Arithmetic!).

Every perfect square,  $P$ , is the product of a positive integer,  $n$ , with itself.

That is,  $P = n \times n$ .

By the Fundamental Theorem of Arithmetic,  $n$  can be written as a product of prime numbers.

Since  $P = n \times n$ , the prime factors of  $P$  are matching pairs of prime factors of  $n$ .

Thus, the prime factors of every perfect square occur an even number of times.

This argument similarly extends to every perfect cube,  $C$ .

Since  $C = n \times n \times n$  for some positive integer  $n$ , then the prime factors of  $C$  occur in sets of three matching prime factors of  $n$ .

Thus for every perfect cube, the number of times that each of the prime factors occurs is a multiple of 3.

3. (a) The first row contains the integers 1 through 6.  
 Each successive row contains the next six integers, in order, that follow the largest integer in the previous row.  
 Thus, the largest integer in any row is six times the row number.  
 Therefore, the largest integer in row 30 is  $6 \times 30 = 180$ .

- (b) By a similar argument to part (a), it follows that the largest integer in row 2012 is  $6 \times 2012 = 12\,072$ .

We find the other numbers in the row by counting backwards.

Thus, the six integers in row 2012 are 12 072, 12 071, 12 070, 12 069, 12 068, 12 067.

The sum of the six integers in row 2012 is

$$12\,072 + 12\,071 + 12\,070 + 12\,069 + 12\,068 + 12\,067 = 72\,417.$$

- (c) Again from part (a), the largest integer in any row is six times the row number.  
 Thus to find the approximate row in which 5000 appears, we divide 5000 by 6.  
 Since  $\frac{5000}{6} = 833\frac{1}{3}$ , and  $6 \times 833 = 4998$ , then the largest integer in row 833 is 4998.  
 Therefore, row 834 contains the next six consecutive integers from 4999 to 5004.  
 (We can check this by recognizing that  $6 \times 834 = 5004$ .)

Thus, the integer 5000 appears in row 834.

Next, we recognize that all even numbered rows list the largest integer in the row beginning in column A through to the smallest integer in column F.

Since row 834 is an even numbered row, then the integers are listed in the order 5004, 5003, 5002, 5001, 5000, 4999, with 5004 beginning in column A.

Therefore, the integer 5000 appears in row 834, column E.

- (d) The largest integer in row  $r$  is  $6 \times r$  or  $6r$ .  
 Since each row contains six consecutive integers, counting backwards the remaining five integers in the row are,  $6r - 1, 6r - 2, 6r - 3, 6r - 4, 6r - 5$ .  
 Thus, the sum of the six integers in row  $r$  is

$$6r + (6r - 1) + (6r - 2) + (6r - 3) + (6r - 4) + (6r - 5) = 36r - 15.$$

Since we require the sum of the six integers in the row to be greater than 10 000, then  $36r - 15 > 10\,000$  or  $36r > 10\,015$  or  $r > \frac{10\,015}{36}$ , and so  $r > 278\frac{7}{36}$ .

But the row number  $r$  must be a whole number, so  $r \geq 279$ .

Since we also require the sum of the six integers in the row to be less than 20 000, then  $36r - 15 < 20\,000$  or  $36r < 20\,015$  or  $r < \frac{20\,015}{36}$ , and so  $r < 555\frac{35}{36}$ .

But the row number  $r$  must be a whole number, so  $r \leq 555$ .

Therefore, the rows in which the six integers have a sum greater than 10 000 and less than 20 000 are 279, 280, 281, ..., 555.

This gives  $555 - 279 + 1$  or 277 rows that satisfy the requirements.

4. (a) The point  $A$  lies on the sphere, vertically above the centre of the sphere,  $O$ .

Similarly, point  $B$  lies on the sphere, vertically below the centre of the sphere.

The top and bottom faces of the cylinder touch the sphere at points  $A$  and  $B$  respectively, as shown.

The segment  $AB$  passes through the centre of the sphere.

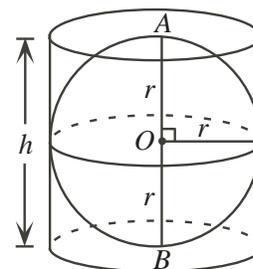
Since  $OA$  is a radius of the sphere, it has length  $r$ .

Similarly,  $OB$  has length  $r$ .

Thus, the length of segment  $AB$  is  $2r$ .

However, segment  $AB$  also represents the perpendicular distance between the top and bottom faces of the cylinder and thus has length equal to the height of the cylinder,  $h$ .

Therefore, the equation relating the height of the cylinder to the radius of the sphere is  $h = 2r$ .



- (b) The volume of the sphere is given by the formula  $\frac{4}{3}\pi r^3$ .

Since the volume of the sphere is  $288\pi$ , then  $\frac{4}{3}\pi r^3 = 288\pi$  or  $4\pi r^3 = 3 \times 288\pi$ , and so  $r^3 = \frac{3 \times 288\pi}{4\pi} = 216$ .

The volume of the cylinder is given by the formula  $\pi r^2 h$ .

From part (a),  $h = 2r$ .

Thus, the volume of the cylinder is  $\pi r^2 h = \pi r^2(2r) = 2\pi r^3$ .

Since  $r^3 = 216$ , then the volume of the cylinder is  $2\pi(216) = 432\pi$ .

- (c) The shape of the space that Darla is able to travel within is determined by the set of points that are exactly 1 km from the nearest point on the cube.

The surface of the cube is comprised of three types of points.

These are points on a face of the cube, points on an edge of the cube, and points at a vertex of the cube.

To determine the volume of space that Darla is able to occupy, we will consider each of these three types of surface points as separate cases.

*Case 1 - Points on a face of the cube.*

The question we must answer is, "What set of points is exactly 1 km away from the nearest point that is on a face of the cube?"

Consider that if Darla begins from any point on a face of the cube, the maximum distance that she can travel is 1 km.

To travel 1 km away from that point, but not be nearer to any other point on the cube, Darla must travel in a direction perpendicular to the face of the cube.

If Darla travels 1 km in a direction perpendicular to the face of a cube, beginning from each of the points on a face of the cube, then the shape of this space that she can occupy is another cube of side length 1 km.

This new cube extends directly outward from the original cube.

Since this can be repeated from each of the 6 faces of the original cube, then in this case Darla can occupy a volume of space equal to  $6 \times 1 \times 1 \times 1$  or  $6 \text{ km}^3$ , as shown in Figure 1.

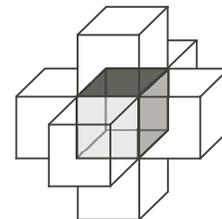


Figure 1

*Case 2 - Points on an edge of the cube.*

The question we must answer is, “What set of points is exactly 1 km away from the nearest point that is on an edge of the cube?”

Consider point  $A$ , the midpoint of the edge on which it lies.

Let points  $B$  and  $C$  be the midpoints of their respective edges also, as shown in Figure 2.

Darla can travel to both points  $B$  and  $C$  since point  $A$  is on the original cube, 1 km away from each of these points.

However, Darla can also travel from  $A$  to any point on the arc  $BC$ . This arc is one quarter of the circumference of the circle with centre  $A$ , radius 1 km, and passing through points  $B$  and  $C$  (since  $\angle BAC = 90^\circ$ ).

Darla can repeat this movement, beginning from any point on this edge. Thus, this shape of space that can be occupied is one quarter of a cylinder of radius 1 km and height 1 km, and has a volume of  $\frac{1}{4}\pi(1)^2(1) = \frac{1}{4}\pi \text{ km}^3$  (see Figure 3).

Since this can be repeated from any point on each of the 12 edges of the original cube, then in this case Darla can occupy a volume of space equal to  $12 \times \frac{1}{4}\pi$  or  $3\pi \text{ km}^3$ , as shown in Figure 4.

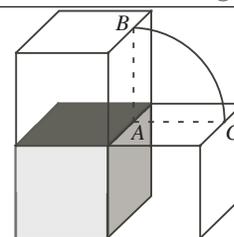


Figure 2

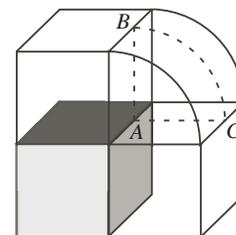


Figure 3

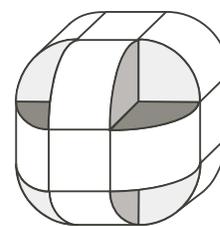


Figure 4

*Case 3 - Points at a vertex of the cube.*

The question we must answer is, “What set of points is exactly 1 km away from the nearest point that is at a vertex of the cube?” Consider point  $P$ , a vertex of the original cube.

Let points  $Q$ ,  $R$  and  $S$  be vertices of external cubes, as shown in Figure 5.

Darla can travel to points  $Q$ ,  $R$  and  $S$  since point  $P$  is on the original cube, 1 km away from each of these points.

From Case 2, we also know that Darla can travel anywhere along the arcs  $QR$ ,  $RS$  and  $SQ$ .

However, Darla can also travel up to 1 km outward from  $P$  to any point on the 3-dimensional surface contained within these 3 arcs.

Since  $\angle SPQ = \angle SPR = \angle QPR = 90^\circ$ , this surface is one eighth of the surface of the sphere with centre  $P$  and radius 1 km (see Figure 5).

Thus, the volume of the space that can be occupied is  $\frac{1}{8} \times \frac{4}{3}\pi(1)^3 = \frac{1}{6}\pi \text{ km}^3$ .

Since this can be repeated from each of the 8 vertices of the original cube, then in this case Darla can occupy a volume of space equal to  $8 \times \frac{1}{6}\pi$  or  $\frac{4}{3}\pi \text{ km}^3$ , as shown in Figure 6.

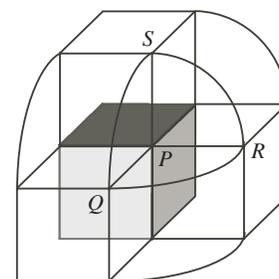


Figure 5

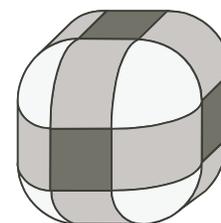
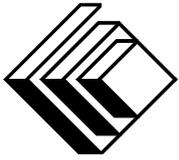


Figure 6

The total volume of space that Darla can occupy is the sum of the volume of space given by the 3 cases above.

That is, Darla can occupy a volume of space equal to  $6 + 3\pi + \frac{4}{3}\pi$  or  $(6 + \frac{13\pi}{3}) \text{ km}^3$ .



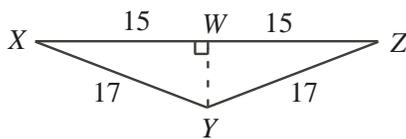
***The CENTRE for EDUCATION in  
MATHEMATICS and COMPUTING***

***2011 Fryer Contest***

**Wednesday, April 13, 2011**

*Solutions*

1. (a) The 5<sup>th</sup> term of the sequence is 14 and the common difference is 3.  
Therefore, the 6<sup>th</sup> term of the sequence is  $14 + 3$  or 17, and the 7<sup>th</sup> term of the sequence is  $17 + 3$  or 20.
  - (b) Each term after the 1<sup>st</sup> is 3 more than the term to its left.  
To get from the 1<sup>st</sup> term to the 31<sup>st</sup> term, we move 30 times to the right.  
Therefore, the 31<sup>st</sup> term is  $30 \times 3$  more than the 1<sup>st</sup> term, or  $2 + 30(3) = 92$ .
  - (c) Using our work from part (b), we want to determine how many times 3 must be added to the first term 2 so that the result is 110.  
That is, how many 3s give  $110 - 2$  or 108.  
Since  $108 \div 3 = 36$ , then 36 3s must be added to 2 to give 110.  
Therefore if the last term is 110, then the number of terms in the sequence is  $36 + 1$  or 37.
  - (d) If 1321 appears in the sequence, then some integer number of 3s added to the first term 2 is equal to 1321.  
That is, we want to determine how many 3s give  $1321 - 2$  or 1319.  
Since 3 does not divide 1319 ( $\frac{1319}{3} = 439\frac{2}{3}$ ), there is no integer number of 3s that when added to the first term 2 is equal to 1321.  
Thus, 1321 does not appear in the sequence.
2. (a) (i) Since  $AB = AC$ , then  $\triangle ABC$  is isosceles.  
Therefore, altitude  $AD$  bisects the base  $BC$  so that  $BD = DC = \frac{14}{2} = 7$ .  
Since  $\angle ADB = 90^\circ$ ,  $\triangle ADB$  is right angled.  
By the Pythagorean Theorem,  $25^2 = AD^2 + 7^2$  or  $AD^2 = 25^2 - 7^2$  or  $AD^2 = 625 - 49 = 576$ , and so  $AD = \sqrt{576} = 24$ , since  $AD > 0$ .
  - (ii) The area of  $\triangle ABC$  is  $\frac{1}{2} \times BC \times AD$  or  $\frac{1}{2} \times 14 \times 24 = 168$ .
  - (b) (i) Through the process described,  $\triangle ADB$  is rotated  $90^\circ$  counter-clockwise about  $D$  to become  $\triangle PDQ$ .  
Similarly,  $\triangle ADC$  is rotated  $90^\circ$  clockwise about  $D$  to become  $\triangle RDQ$ .  
Through both rotations, the lengths of the sides of the original triangles remain unchanged.  
Thus,  $PD = AD = 24$  and  $RD = AD = 24$ .  
Since  $P$ ,  $D$  and  $R$  lie in a straight line, then base  $PR = PD + RD = 24 + 24 = 48$ .
  - (ii) *Solution 1*  
When  $\triangle ADC$  is rotated  $90^\circ$  clockwise about  $D$ , side  $DC$  becomes altitude  $DQ$  in  $\triangle PQR$ .  
Therefore,  $DQ = DC = 7$ .  
Thus, the area of  $\triangle PQR$  is  $\frac{1}{2} \times PR \times DQ$  or  $\frac{1}{2} \times 48 \times 7 = 168$ .
  - Solution 2*  
As determined in part (a), the area of  $\triangle ABC = 168$ .  
The area of  $\triangle ABC$  is equal to the area of  $\triangle ADB$  added to the area of  $\triangle ADC$ .  
Thus, the sum of the areas of these two smaller triangles is also 168.  
Through the rotations described, triangles  $ADB$  and  $ADC$  remain unchanged.  
Since  $\triangle PDQ$  is congruent to  $\triangle ADB$ , their areas are equal.  
Since  $\triangle RDQ$  is congruent to  $\triangle ADC$ , their areas are equal.  
Therefore, the area of  $\triangle PQR$  is equal to the sum of the areas of triangles  $PDQ$  and  $RDQ$ , which equals the sum of the areas of triangles  $ADB$  and  $ADC$  or 168.
  - (c) Since  $XY = YZ$ , then  $\triangle XYZ$  is isosceles.  
Draw altitude  $YW$  from  $Y$  to  $W$  on  $XZ$ .  
Altitude  $YW$  bisects the base  $XZ$  so that  $XW = WZ = \frac{30}{2} = 15$ , as shown.



Since  $\angle YWX = 90^\circ$ ,  $\triangle YWX$  is right angled.

By the Pythagorean Theorem,  $17^2 = YW^2 + 15^2$  or  $YW^2 = 17^2 - 15^2$  or  $YW^2 = 289 - 225 = 64$ , and so  $YW = \sqrt{64} = 8$ , since  $YW > 0$ .

By reversing the process described in part (b), we rotate  $\triangle XWY$  clockwise  $90^\circ$  about  $W$  and similarly rotate  $\triangle ZWY$  counter-clockwise  $90^\circ$  about  $W$  to obtain a new isosceles triangle with the same area.

The new triangle formed has two equal sides of length 17 (since  $XY$  and  $ZY$  form these sides) and a third side having length twice that of  $YW$  or  $2 \times 8 = 16$  (since the new base consists of two copies of  $YW$ ).

3. (a)

Two-digit number	Step 1	Step 2	Step 3
68	$6 \times 8 = 48$	$4 \times 8 = 32$	$3 \times 2 = 6$

Beginning with 68, the process stops after 3 steps.

- (b) For the process to stop at 8, the preceding number must have two digits whose product is 8.

The factor pairs of 8 are:  $1 \times 8$ ;  $2 \times 4$ .

Thus, the only two-digit numbers whose digits have a product of 8 are 18, 81, 24, and 42.

Next, consider the factor pairs of 18, 81, 24, and 42.

The factor pairs of 18 are:  $1 \times 18$ ;  $2 \times 9$ ;  $3 \times 6$ .

Since the factors 1 and 18 cannot be used to form a two-digit number, the only two-digit numbers whose digits have a product of 18 are 29, 92, 36, 63.

The factor pairs of 81 are:  $1 \times 81$ ;  $3 \times 27$ ;  $9 \times 9$ .

Thus, the only two-digit number whose digits have a product of 81 is 99.

The factor pairs of 24 are:  $1 \times 24$ ;  $2 \times 12$ ;  $3 \times 8$ ;  $4 \times 6$ .

Thus, the only two-digit numbers whose digits have a product of 24 are 38, 83, 46, 64.

The factor pairs of 42 are:  $1 \times 42$ ;  $3 \times 14$ ;  $6 \times 7$ .

Thus, the only two-digit numbers whose digits have a product of 42 are 67, 76.

A summary is shown in the table below.

Two-digit number	Preceding number
18	29, 92, 36, 63
81	99
24	38, 83, 46, 64
42	67, 76

Therefore, the only two digit numbers for which the process stops at 8 after 2 steps are 29, 92, 36, 63, 99, 38, 83, 46, 64, 67, 76.

- (c) For the process to stop at 4, the preceding number must have two digits whose product is 4.

The factor pairs of 4 are:  $1 \times 4$ ;  $2 \times 2$ .

The only two-digit numbers whose digits have a product of 4 are 14, 41, and 22.

We now find all two-digit numbers whose digits have a product of 14, 41, or 22.

Next, consider the factor pairs of 14, 41 and 22.

The factor pairs of 14 are:  $1 \times 14$ ;  $2 \times 7$ .

The only two-digit numbers whose digits have a product of 14 are 27 and 72.

The only factor pair of 41 is  $1 \times 41$ .

Since 1 and 41 cannot be used to form a two-digit number, there are no two-digit numbers whose digits have a product of 41.

The factor pairs of 22 are:  $1 \times 22$ ;  $2 \times 11$ .

Thus, there are no two-digit numbers whose digits have a product of 22.

A summary is shown in the table below.

Two-digit number	Preceding number
14	27, 72
41	there are none
22	there are none

We now find all two-digit numbers whose digits have a product of 27 or 72.

This is summarized in the table below.

Two-digit number	Preceding number
27	39, 93
72	89, 98

Since there are no two-digit numbers whose digits have a product of 39, 93, 89 or 98, we have completed our list.

The only two digit numbers for which the process stops at 4 are 14, 41, 22, 27, 72, 39, 93, 89, 98.

- (d) Although systematic trial and error will find us *the* two-digit number for which the process stops after 4 steps (there is only one), some work done earlier in this question may save us some time.

Notice in part (b) that there are 11 different numbers for which the process stops at 8 after 2 steps.

Of these 11 numbers only 3 can be continued on backwards to previous steps.

These are 36, 63 and 64.

Since there is only one step that precedes 63, that is 79 or 97, we have not found a 4-step number.

Since there is only one step that precedes 64, that is 88, we have not found a 4-step number.

However, 49 precedes 36 and 77 precedes 49.

That is, beginning with 77 the process stops after 4 steps as shown below.

Two-digit number	Step 1	Step 2	Step 3	Step 4
77	$7 \times 7 = 49$	$4 \times 9 = 36$	$3 \times 6 = 18$	$1 \times 8 = 8$

Therefore, 77 is a two-digit number for which the process stops after 4 steps.

Can you show that 77 is the only two-digit number for which the process stops after 4 steps?

Can you show that there is no two-digit number for which the process stops after more than 4 steps?

4. (a) Initially, Ian has 365 two-dollar coins.  
That is, Ian begins with  $365 \times \$2$  or \$730.  
After 365 days, he has spent  $365 \times \$1.72$  or \$627.80.  
Therefore, Ian will have  $\$730 - \$627.80$  or \$102.20 remaining in the coin jar after 365 days.
- (b) Since Ian starts with 365 \$2.00 coins and his tea costs \$1.72 each day for 365 days, then he will use at most one of the \$2.00 coins each day. This means that he always has \$2.00 coins left.  
Since Ian pays with the least amount possible each day, then he never pays more than \$2.00; he will pay with less than \$2.00 if he has accumulated at least \$1.72 in “loose change” (that is, in coins valued less than \$2.00).  
Since Ian pays each time with \$2.00 or less, then he receives at most  $\$2.00 - \$1.72 = \$0.28$  in change each time.  
Since Ian receives at most \$0.28 in change each time, then he receives at most one 25¢ coin each time.  
We will show that the maximum number of 25¢ coins that he can have is 7.  
Suppose Ian had 8 or more 25¢ coins in his jar at one time.  
In this case, he must have had 7 or more 25¢ coins in his jar on the previous day since he never gets more than one 25¢ coin back in change.  
But if Ian had 7 or more 25¢ coins in his jar, then he has at least \$1.75 in loose change and so would give the cashier at most \$1.75 and so would get at most \$0.03 in change.  
In this case, he would not get a 25¢ coin as change and so would never get to 8 25¢ coins.  
Therefore, Ian can never have more than 7 25¢ coins in his jar.  
Lastly, we show that the maximum number of 25¢ coins is indeed 7 by showing that this actually happens.  
Note that when Ian gives the cashier a \$2.00 coin and receives \$0.28 in change, the change will come as 1 25¢ coin and 3 1¢ coins.  
Starting on the first day, Ian gives the cashier a \$2.00 coin and receives 28¢ in loose change until he has accumulated more than \$1.72 in loose change.  
Note that after 6 days, Ian will have received  $6 \times \$0.28 = \$1.68$  in loose change (6 25¢ coins and  $3 \times 6 = 18$  1¢ coins).  
Since Ian still has less than \$1.72, he must again use a \$2.00 coin on day 7, and so he again receives 28¢ in loose change on day 7, giving him 7 25¢ coins and 21 1¢ coins in total.  
Therefore, there is a point at which Ian has 7 25¢ coins and Ian can never have 8 25¢ coins.  
Thus, the maximum number of 25¢ coins that he can have is 7.
- (c) Part (b) describes Ian giving the cashier a \$2.00 coin and receiving \$0.28 in change, the change coming as 1 25¢ coin and 3 1¢ coins.  
This situation happens frequently, so we define it to be a *typical day*.  
Typical days occur whenever Ian has less than \$1.72 in loose change and he must offer the cashier a \$2.00 coin as payment for the tea.  
For each day listed in the table below, the number of each type of coin in the jar at the end of the day is given.  
To condense the table, some typical days have been omitted.  
However, on each of these typical days we know that Ian paid with a \$2.00 coin and received 1 25¢ coin and 3 1¢ coins as change.

Coin Type	Day 7	Day 8	Day 14	Day 15	Day 21	Day 22	Day 28	Day 29	Day 35	Day 36	Day 42	Day 43	Day 49	Day 50
\$2.00	358	358	352	352	346	346	340	340	334	334	328	328	322	322
25¢	7	0	6	0	6	0	6	0	6	0	6	0	6	0
1¢	21	24	42	20	38	16	34	12	30	8	26	4	22	0

To clarify the table somewhat, the omitted days 9 to 13 (for example) are typical days. That is, Ian will offer a \$2.00 coin as payment and the number of \$2.00 coins in the jar will decrease by 1 on each of these days.

Ian will receive exactly 1 25¢ coin and 3 1¢ coins as change on each of these days.

The table shows that beginning at day 8, every 7<sup>th</sup> day is not a typical day.

That is, on days 8, 15, 22, 29, 36, 43 and 49, Ian has at least \$1.72 in loose change that he offers the cashier for payment instead of using a \$2.00 coin.

All other days listed in the table and omitted from the table are typical days.

After 50 days, Ian has used exactly  $365 - 322 = 43$  \$2.00 coins and has only \$2.00 coins remaining in the coin jar.

This is the first day that the coin jar returns to its initial state of containing only \$2.00 coins.

Since we are starting again on day 51 with only \$2.00 coins, the cycle of coins offered by Ian and received from the cashier will repeat every 50 days.

That is, after 250 days Ian will have used exactly  $5 \times 43 = 215$  \$2.00 coins and the coin jar will contain  $365 - 215 = 150$  \$2.00 coins only (there will be no 25¢ or 1¢ coins).

Thus, the number of 25¢ and 1¢ coins contained in the jar on the 277<sup>th</sup> day will be the same as those in the jar on the 27<sup>th</sup> day.

Using the table above, the number of 25¢ and 1¢ coins on the 28<sup>th</sup> day is 6 and 34, respectively.

Therefore, the number of 25¢ and 1¢ coins on the 27<sup>th</sup> day is 5 and 31, respectively, since the 28<sup>th</sup> day is a typical day.

We also note that there are 341 \$2.00 coins in the jar on the 27<sup>th</sup> day.

Thus, Ian used  $365 - 341 = 24$  \$2.00 coins in the first 27 days.

As a result of the repeating pattern of coins in the jar, 24 \$2.00 coins will be used from the 251<sup>st</sup> day to the end of the 277<sup>th</sup> day.

Since there are 150 \$2.00 coins in the jar after 250 days, then there are  $150 - 24 = 126$  \$2.00 coins in the jar after 277 days.

After 277 days, there will be 126 \$2.00 coins, 5 25¢ coins, and 31 1¢ coins in the jar.

We can check if our final answer is reasonable.

After 277 days, Ian has spent  $277 \times \$1.72 = \$476.44$ .

Therefore Ian should have  $\$730 - \$476.44 = \$253.56$  remaining in the coin jar.

From the answer given, the coin jar contains  $126 \times \$2.00 + 5 \times 25¢ + 31¢ = \$253.56$ .

We can be reasonably confident that we have answered this question correctly.

It is worth noting that Ian initially has \$2.00 coins or 200¢ coins at his disposal and the price of each tea is 172¢.

The lowest common multiple of 200 and 172 is 8600 (verify this for yourself).

Can you explain how we may have used this lowest common multiple to find the number of days it takes for the jar to return to its initial state of containing only \$2.00 coins?



**Canadian  
Mathematics  
Competition**

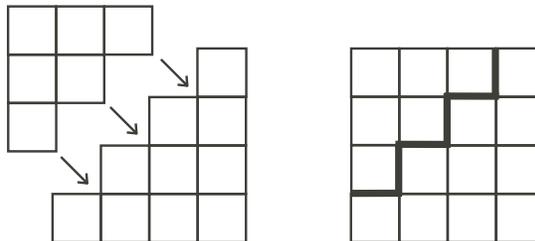
*An activity of the Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario*

***2010 Fryer Contest***

**Friday, April 9, 2010**

*Solutions*

1. (a) The piece on the right can be repositioned to rest on the left piece as shown. The resulting arrangement is a 4 by 4 square having 16 tiles.

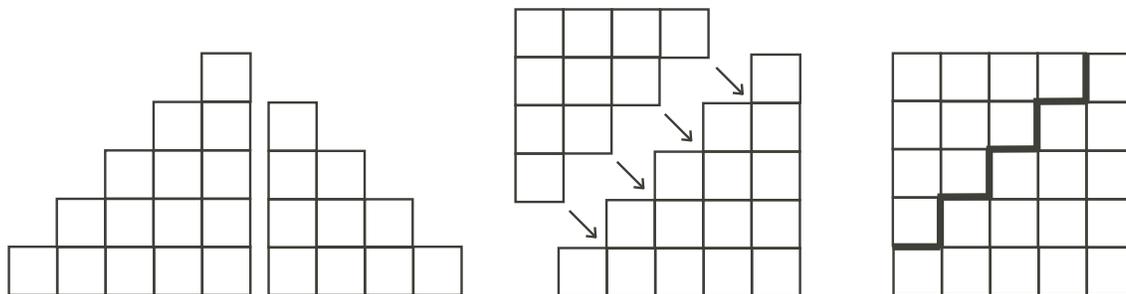


- (b) *Solution 1*

The top 4 rows of Figure 5 are identical to the 4 rows that form Figure 4. The bottom row of Figure 5 has two tiles more than the bottom row of Figure 4, or  $7 + 2 = 9$  tiles. Thus, Figure 5 has  $1 + 3 + 5 + 7 + 9 = 25$  tiles.

*Solution 2*

The bottom (5th) row of Figure 5 has two tiles more than the bottom row of Figure 4, or  $7 + 2 = 9$  tiles. Using the method of part (a), Figure 5 can be cut into two pieces and repositioned as shown. The resulting arrangement is a 5 by 5 square having 25 tiles.



- (c) We count the number of tiles in the bottom row of each of the first 5 figures and list the results in the following table:

Figure Number	Number of tiles in the bottom row
1	1
2	3
3	5
4	7
5	9

Since the number of tiles in the bottom row of any figure is two more than the number of tiles in the bottom row of the previous figure, we can continue the table.

Figure Number	Number of tiles in the bottom row
6	11
7	13
8	15
9	17
10	19

Therefore, Figure 10 has 19 tiles in its bottom row.

(d) *Solution 1*

The top 9 rows of Figure 11 are identical to the 9 rows that form Figure 9.

Since Figure 11 has 11 rows, the difference between the total number of tiles in Figure 11 and the total number of tiles in Figure 9 is equal to the sum of the number of tiles in the 10th and 11th rows of Figure 11.

The number of tiles in the 10th row of Figure 11 is equal to the number of tiles in the 10th (bottom) row of Figure 10, or 19.

From part (c), we also know that the bottom row of Figure 11 has 2 tiles more than the bottom row of Figure 10, or  $19 + 2 = 21$  tiles.

Therefore, the difference between the total number of tiles in Figure 11 and the total number of tiles in Figure 9 is  $19 + 21 = 40$  tiles.

*Solution 2*

Using the method of part (a), it can be shown that Figure 11 can be cut into 2 pieces and rearranged to form an 11 by 11 square having  $11 \times 11 = 121$  tiles.

Similarly, Figure 9 can be shown to have  $9 \times 9 = 81$  tiles.

Therefore, the difference between the total number of tiles in Figure 11 and the total number of tiles in Figure 9 is  $121 - 81 = 40$  tiles.

2. (a) The average of any set of integers is equal to the sum of the integers divided by the number of integers.

Thus, the average of the integers is  $\frac{71+72+73+74+75}{5} = \frac{365}{5} = 73$ .

- (b) (i) Simplifying,  $n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10$ .

(ii) Since the sum of the 5 consecutive integers is  $5n + 10$ , the average of these integers is  $\frac{5n + 10}{5} = n + 2$ .

If  $n$  is an even integer, then  $n + 2$  is an even integer.

If  $n$  is an odd integer, then  $n + 2$  is an odd integer.

Thus for the average  $n + 2$  to be odd, the integer  $n$  must be odd.

- (c) Simplifying,  $n + (n + 1) + (n + 2) + (n + 3) + (n + 4) + (n + 5) = 6n + 15$ .

Since the sum of the 6 consecutive integers is  $6n + 15$ , the average of these integers is  $\frac{6n + 15}{6} = n + \frac{15}{6} = n + \frac{5}{2}$ .

For every integer  $n$ ,  $n + \frac{5}{2}$  is never an integer.

Therefore, the average of six consecutive integers is never an integer.

3. (a) Travelling at a speed of 60 km/h for 9 hours, Train 1 travels a distance of  $60 \times 9 = 540$  km.

If the distance from Amville to Batton is  $d$ , then  $\frac{2}{3}d = 540$  and  $d = \frac{540(3)}{2} = 810$  km.

Therefore, the distance from Amville to Batton is 810 km.

- (b) The distance from Batton to Amville is 810 km.

Two-thirds of the distance from Batton to Amville is  $\frac{2}{3} \times 810 = 540$  km.

Train 2 travelled the 540 km in 6 hours.

Thus, Train 2 travels at a constant speed of  $\frac{540}{6} = 90$  km/h.

- (c) Let  $t$  hours be the time that it takes Train 1 to travel from Amville to Cuford.

Since Train 1 travels at 60 km/h, the distance from Amville to Cuford is  $60t$  km.

Train 2 leaves Batton  $3\frac{1}{2}$  hours after Train 1 leaves Amville.

Since the trains arrive at Cuford at the same time, Train 2 takes  $(t - 3\frac{1}{2})$  hours to travel from Batton to Cuford.

Since Train 2 travels at 90 km/h, the distance from Batton to Cuford is  $90(t - 3\frac{1}{2})$  km. The distance from Amville to Cuford added to the distance from Cuford to Batton is equal to the distance from Amville to Batton, 810 km.

Thus,  $60t + 90(t - 3\frac{1}{2}) = 810$  or  $60t + 90t - 315 = 810$  or  $150t = 1125$  and  $t = 7\frac{1}{2}$  hours. Therefore, Train 1 needed  $7\frac{1}{2}$  hours to travel from Amville to Cuford.

Since Train 1 arrived at Cuford at 9:00 p.m., Train 1 left Amville at 1:30 p.m.

4. (a) A palindrome less than 1000 must be either 1, 2 or 3 digits in length.

We consider each of these 3 possibilities as separate cases below.

Case 1: one-digit palindromes

All of the positive integers from 1 to 9 are palindromes since they are the same when read forwards or backwards. Thus, there are 9 one-digit palindromes.

Case 2: two-digit palindromes

We are looking for all of the palindromes from the positive integers 10 to 99.

To be a two-digit palindrome, the first digit must equal the second digit.

Thus, there are only 9 two-digit palindromes;  $\{11, 22, 33, 44, 55, 66, 77, 88, 99\}$ .

Case 3: three-digit palindromes

All three-digit palindromes are of the form  $aba$ , for integers  $a$  and  $b$ , where  $1 \leq a \leq 9$  and  $0 \leq b \leq 9$ .

That is, to create a three-digit palindrome the hundreds digit and the units digit must be equal to each other, but not equal to zero.

Since the hundreds digit  $a$  can be any of the positive integers from 1 to 9, there are 9 choices for the hundreds digit  $a$ .

Since the tens digit  $b$  can be any of the integers from 0 to 9, there are 10 choices for  $b$ .

The final digit (the units digit) must equal the hundreds digit, thus there is no choice for the units digit since it has already been chosen.

In total, there are  $9 \times 10$  choices for  $a$  and  $b$  and therefore, 90 three-digit palindromes.

There are 9 one-digit palindromes, 9 two-digit palindromes and 90 three-digit palindromes.

In total, there are  $9 + 9 + 90 = 108$  palindromes less than 1000.

- (b) All seven-digit palindromes are of the form  $abcdcba$ , for integers  $a, b, c$ , and  $d$ , where  $1 \leq a \leq 9$  and  $0 \leq b, c, d \leq 9$ .

Since the last 3 digits  $cba$  are determined by the first three digits  $abc$ , the number of seven-digit palindromes is dependent on the number of choices for the first 4 digits  $abcd$  only. There are 9 choices for the first digit  $a$ , 10 choices for the second digit  $b$ , 10 choices for the third digit  $c$ , and 10 choices for the fourth digit  $d$ .

In total, there are  $9 \times 10 \times 10 \times 10$  choices for  $abcd$  and therefore, 9000 seven-digit palindromes.

- (c) The seven-digit palindromes between 1000000 and 2000000 begin with a 1, and are of the form  $1bcdcb1$ .

As in part (b), there are 10 choices for each of  $b, c$  and  $d$ , and therefore  $10 \times 10 \times 10 = 1000$  palindromes between 1000000 and 2000000.

In the increasing list of seven-digit palindromes, these are the first 1000 palindromes.

Similarly, the next 1000 palindromes lie between 2000000 and 3000000, and the next 1000 lie between 3000000 and 4000000.

Thus, the 2125th palindrome must lie between 3000000 and 4000000 and must be of the form  $3bcdcb3$ .

The smallest palindromes of the form  $3bcdcb3$  are those of the form  $30cdc03$ .

Since there are 10 possibilities for each of  $c$  and  $d$ , then there are  $10 \times 10 = 100$  such

palindromes.

The largest of these (3099903) is the 2100th palindrome in the list.

Similarly, there are 100 palindromes of the form  $31cdc13$ .

The 2125th palindrome will thus be of this form.

The smallest palindromes of the form  $31cdc13$  are those of the form  $310d013$ .

Since there are 10 possibilities for  $d$ , then there are 10 such palindromes.

Similarly, there are 10 palindromes of the form  $311d113$ .

The largest of these (3119113) is the 2120th in the list.

Also, there are 10 palindromes of the form  $312d213$ .

The 2125th palindrome in the entire list is the 5th largest of these, or  $3124213$ .

- (d) All six-digit palindromes are of the form  $abccba$ , for integers  $a, b, c$ , where  $1 \leq a \leq 9$  and  $0 \leq b, c \leq 9$ .

The integer  $N$  formed by these digits  $abccba$  can be expressed by considering the place value of each of its six digits.

That is,  $N = 100000a + 10000b + 1000c + 100c + 10b + a = 100001a + 10010b + 1100c$ .

Recognize that the integers 100001, 10010 and 1100 have a common factor of 11, so then  $N = 11(9091a + 910b + 100c)$  or  $N = 11k$  where  $k = 9091a + 910b + 100c$ .

For  $N$  to be divisible by 91, the integer  $11k$  must be divisible by 91.

Since 11 is a prime number and not a divisor of 91, then  $N$  is divisible by 91 if

$k = 9091a + 910b + 100c$  is divisible by 91. We rewrite  $k$  as,

$$\begin{aligned} k &= 9091a + 910b + 100c \\ &= (9100a - 9a) + 910b + (91c + 9c) \\ &= (9100a + 910b + 91c) + (9c - 9a) \\ &= 91(100a + 10b + c) + 9(c - a) \end{aligned}$$

Since  $91(100a + 10b + c)$  is divisible by 91 for any choices of  $a, b, c$ , then  $k$  is divisible by 91 only if  $9(c - a)$  is divisible by 91.

Since 9 has no factors in common with 91,  $9(c - a)$  is only divisible by 91 if  $(c - a)$  is divisible by 91.

Recall that  $1 \leq a \leq 9$  and  $0 \leq c \leq 9$ , thus 91 divides  $(c - a)$  only if  $c - a = 0$  or  $c = a$ .

Therefore, the six-digit palindromes that are divisible by 91 are of the form  $abaaba$  where  $1 \leq a \leq 9$  and  $0 \leq b \leq 9$ .

Since there are 9 choices for  $a$  and 10 choices for  $b$ , there are a total of  $9 \times 10 = 90$  six-digit palindromes that are divisible by 91.



**Canadian  
Mathematics  
Competition**

*An activity of the Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario*

***2009 Fryer Contest***

**Wednesday, April 8, 2009**

*Solutions*

1. (a) The total cost to make 100 cups of lemonade is  
 $\$0.15 \times 100 + \$12.00 = \$15.00 + \$12.00 = \$27.00$ .
- (b) The total money earned for selling 100 cups of lemonade is  $\$0.75 \times 100 = \$75.00$ .  
 The profit is the total money earned minus the total cost, or  $\$75.00 - \$27.00 = \$48.00$ .
- (c) Let  $x$  represent the number of cups that Emily must sell to break even.  
 The total cost to make  $x$  cups of lemonade is  $\$0.15 \times x + \$12.00 = \$0.15x + \$12.00$ .  
 The total money earned by selling  $x$  cups is  $\$0.75 \times x$  or  $\$0.75x$ .  
 For a profit of \$0, total cost must equal total money earned.  
 Therefore,  $\$0.15x + \$12.00 = \$0.75x$  or  $\$12.00 = \$0.60x$  or  $x = 20$ .  
 Emily must sell 20 cups of lemonade to break even.
- (d) Let  $n$  represent the number of cups that Emily must sell to make a profit of exactly \$17.00.  
 As in (c), the total cost to make  $n$  cups of lemonade is  $\$0.15n + \$12.00$ .  
 As in (c), the total money earned by selling  $n$  cups is  $\$0.75n$ .  
 For a profit of \$17.00, the total money earned minus the total cost must equal \$17.00.  
 Thus,  $\$0.75n - (\$0.15n + \$12.00) = \$17.00$  or  $\$0.60n - \$12.00 = \$17.00$  or  $\$0.60n = \$29.00$ .  
 This gives  $n = \frac{29}{0.6} = \frac{290}{6} = 48\frac{1}{3}$ .  
 Since  $n$  represents the number of cups that Emily sells,  $n$  must be a non-negative integer and as such, cannot equal  $48\frac{1}{3}$ .  
 Therefore, it is not possible for Emily to make a profit of exactly \$17.00.

2. (a) Evaluating,  $2\nabla 5 = \frac{2+5}{1+2 \times 5} = \frac{7}{11}$ .

- (b) Evaluating the expression in brackets first,

$$(1\nabla 2)\nabla 3 = \left( \frac{1+2}{1+1 \times 2} \right) \nabla 3 = \left( \frac{3}{3} \right) \nabla 3 = 1\nabla 3 = \frac{1+3}{1+1 \times 3} = 1.$$

(Note that for any  $b > 0$ ,  $1\nabla b = \frac{1+b}{1+1 \times b} = \frac{1+b}{1+b} = 1$ .)

- (c) By definition,  $2\nabla x = \frac{2+x}{1+2x}$ . Thus,

$$\begin{aligned} \frac{2+x}{1+2x} &= \frac{5}{7} \\ 7(2+x) &= 5(1+2x) \\ 14+7x &= 5+10x \\ 9 &= 3x \end{aligned}$$

so  $x = 3$ .

- (d) We have,  $x\nabla y = \frac{x+y}{1+xy}$ . Thus, from the given information,  $\frac{x+y}{1+xy} = \frac{x+y}{17}$ .

The numerator,  $x+y$ , of each of these two fractions is non-zero since  $x > 0$  and  $y > 0$ .

Two equivalent fractions having equal, non-zero numerators have equal denominators.

Thus,  $1+xy = 17$  or  $xy = 16$ . The possible ordered pairs of positive integers  $(x, y)$ , for which  $xy = 16$ , are  $(1, 16)$ ,  $(16, 1)$ ,  $(2, 8)$ ,  $(8, 2)$ , and  $(4, 4)$ .

3. (a) We know that  $OA$  and  $OB$  are each radii of the semi-circle with centre  $O$ .  
 Thus,  $OA = OB = OC + CB = 32 + 36 = 68$ .  
 Therefore,  $AC = AO + OC = 68 + 32 = 100$ .
- (b) The semi-circle with centre  $K$  has radius  $AK = \frac{1}{2}(AC) = \frac{1}{2}(100) = 50$ .  
 Thus, this semi-circle has an area equal to  $\frac{1}{2}\pi(AK)^2 = \frac{1}{2}\pi(50)^2 = 1250\pi$ .

- (c) The shaded area is equal to the area of the largest semi-circle with centre  $O$ , minus the combined areas of the two smaller unshaded semi-circles with centres  $K$  and  $M$ .  
 The radius of the smaller unshaded circle is  $MB = \frac{1}{2}(CB) = \frac{1}{2}(36) = 18$ .  
 Therefore, the shaded area equals

$$\begin{aligned} & \frac{1}{2}\pi(OB)^2 - \left(\frac{1}{2}\pi(AK)^2 + \frac{1}{2}\pi(MB)^2\right) \\ &= \frac{1}{2}\pi(68)^2 - \left(\frac{1}{2}\pi(50)^2 + \frac{1}{2}\pi(18)^2\right) \\ &= \frac{1}{2}\pi(68^2 - 50^2 - 18^2) \\ &= \frac{1}{2}\pi(4624 - 2500 - 324) \\ &= \frac{1}{2}\pi(1800) \\ &= 900\pi \end{aligned}$$

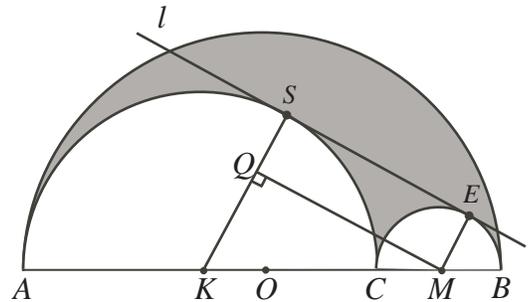
- (d) *Solution 1*

Construct line segments  $KS$  and  $ME$  perpendicular to line  $l$ .

Position point  $Q$  on  $KS$  so that  $MQ$  is perpendicular to  $KS$ , as shown.

In quadrilateral  $MQSE$ ,  
 $\angle MQS = \angle QSE = \angle SEM = 90^\circ$ .

Hence, quadrilateral  $MQSE$  is a rectangle.



The larger unshaded semi-circle has radius 50, so  $KC = KS = 50$ .

The smaller unshaded semi-circle has radius 18, so  $ME = MC = MB = 18$ .

Thus,  $MK = MC + KC = 18 + 50 = 68$ .

The area of quadrilateral  $KSEM$  is the sum of the areas of rectangle  $MQSE$  and  $\triangle MKQ$ .  
 Since  $QS = ME = 18$ , then  $KQ = KS - QS = 50 - 18 = 32$ .

Using the Pythagorean Theorem in  $\triangle MKQ$ ,  $MK^2 = KQ^2 + QM^2$  or  $68^2 = 32^2 + QM^2$   
 or  $QM = \sqrt{68^2 - 32^2} = 60$  (since  $QM > 0$ ).

The area of  $\triangle MKQ$  is  $\frac{1}{2}(KQ)(QM) = \frac{1}{2}(32)(60) = 960$ .

The area of rectangle  $MQSE$  is  $(QM)(QS) = (60)(18) = 1080$ .

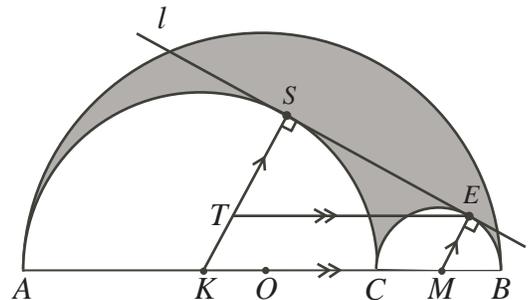
Thus, the area of quadrilateral  $KSEM$  is  $960 + 1080 = 2040$ .

*Solution 2*

Construct line segments  $KS$  and  $ME$  perpendicular to line  $l$ .

Since  $KS$  and  $ME$  are each perpendicular to line  $l$ , they are parallel to one another and thus,  $KSEM$  is a trapezoid.

Position point  $T$  on  $KS$  so that  $TE$  is parallel to  $KM$ , as shown.



The larger unshaded semi-circle has radius 50, so  $KC = KS = 50$ .

The smaller unshaded semi-circle has radius 18, so  $ME = MC = MB = 18$ .

Thus,  $KM = KC + CM = 50 + 18 = 68$ .

Since  $KTEM$  is a parallelogram  $TE = KM = 68$  and  $KT = ME = 18$ .

Thus,  $TS = KS - KT = 50 - 18 = 32$ .

Using the Pythagorean Theorem in  $\triangle TSE$ ,  $SE^2 = TE^2 - TS^2$  or  $SE^2 = 68^2 - 32^2$  or  
 $SE = \sqrt{68^2 - 32^2} = 60$  (since  $SE > 0$ ).

The area of trapezoid  $KSEM$  is  $\frac{(SE)(ME + KS)}{2} = \frac{(60)(18 + 50)}{2} = (30)(68) = 2040$ .

4. When we are calculating such a sum by hand, in each column starting at the right hand end, we add up the digits and add the carry to this to create a column total. We take this total, write down its unit digit and take the integer formed by the rest of the total to become the carry into the next column to the left.

For example, if a column total is 124, we write down the units digit 4 beneath this column and carry 12 into the next column to the left.

- (a) The sum of the digits in the ones column is  $2(101) = 202$ . Thus, the ones digit  $A$  is 2, and 20 is carried left to the tens column.
- (b) The sum of the digits in the tens column is  $2(100) = 200$  and the carry into the tens column is 20. Thus, the total in the tens column is  $200 + 20 = 220$ . Therefore, the tens digit  $B$  is 0 and 22 is carried left to the hundreds column. The sum of the digits in the hundreds column is  $2(99) = 198$ , which when added to the carry of 22 gives a column total of  $198 + 22 = 220$ . Thus, the hundreds digit  $C$  is also 0.
- (c) We show that the middle digit of the sum is 3 by using the following steps:
- Step 1: The sum has 101 digits
  - Step 2: The total in the middle column is at least 113
  - Step 3: The total in the middle column cannot be 114 or more

Steps 2 and 3 together will tell us that the total in the middle column is exactly 113, so the middle digit in the sum is the units digit of 113, which is 3. We will need to use the following Fact:

Fact: The carry from one column to the next is always at most 22.

We leave the explanation of this Fact to the end, but use it twice. We number the columns in the sum from left to right.

Step 1: The sum has 101 digits

The total in the 1st column is 2 plus the carry from the 2nd column. Thus, the total in the 1st column will have only 1 digit unless the carry from the 2nd column is at least 8. For the carry from the 2nd column to be at least 8, then the total in the 2nd column must be at least 80. The sum of the digits in the 2nd column is 4, so for a total of at least 80, the carry from the 3rd column would have to be at least 76.

But every carry is at most 22 (see the Fact below), so this is impossible.

Thus, the total in the 1st column has only 1 digit.

This means that the sum has exactly the same number of digits as the largest number among the numbers being added, so has 101 digits.

Step 2: The total in the middle column is at least 113

Since the sum has 101 digits, then the middle digit (or column) is the 51st column from the left. (There will be 50 digits to the left of this column and 50 digits to the right, giving 101 digits in total.)

The sum of the digits in the 51st column is  $2(51) = 102$ .

The sum of the digits in the 52nd column is  $2(52) = 104$ .

The sum of the digits in the 53rd column is  $2(53) = 106$ .

Thus, the carry from the 53rd column to the 52nd column is at least 10, so the total of the 52nd column is at least  $104 + 10 = 114$ .

Thus, the carry from the 52nd column to the 51st column is at least 11, so the total of the 51st column is at least  $102 + 11 = 113$ .

Step 3: The total in the middle column cannot be 114 or more

If the total in the 51st column was 114 or more, then the carry from the 52nd column would have to be at least  $114 - 102 = 12$ .

For the carry from the 52nd column to be at least 12, then the total in the 52nd column would have to be at least 120. For this total to be at least 120, then the carry from the 53rd column would have to be at least  $120 - 104 = 16$ .

For the carry from the 53rd column to be at least 16, then the total in the 53rd column would have to be at least 160. For this total to be at least 160, then the carry from the 54th column would have to be at least  $160 - 106 = 54$ .

But by the Fact (see below), the carry cannot be more than 22, so this is impossible.

Therefore, the total in the middle column cannot be 114 or more, so must be exactly 113.

It remains to look at the Fact.

Fact: The carry from one column to the next is always at most 22

We start from the rightmost column.

The sum of the digits in the 101st column is  $2(101) = 202$ , so 20 is carried to the 100th column.

The total of the 100th column is thus  $2(100) + 20 = 220$ , so 22 is carried to the 99th column.

The total of the 99th column is thus  $2(99) + 22 = 220$ , so 22 is carried to the 98th column. To this point, none of the carries is larger than 22.

Suppose that at some point the carry is at least 23.

If this is the case, then starting from the right end, there must be a first time that the carry is at least 23.

Let's suppose that this first time is the carry from  $n$ th column.

We know that  $n \leq 98$ , since the carries from the first three columns are all at most 22.

Also, we know that the carry into the  $n$ th column is at most 22, since the carry from the  $n$ th column is the first time that the carry is more than 22.

Let's look at this  $n$ th column.

The  $n$ th column includes  $n$  2's, so the sum of the digits in this column is  $2n$ .

Since  $n \leq 98$ , then the sum of the digits in the  $n$ th column is at most  $2(98) = 196$ .

For the carry from the  $n$ th column to be at least 23, the total in the  $n$ th column must be at least 230, which means that the carry into the  $n$ th column is at least  $230 - 196 = 34$ .

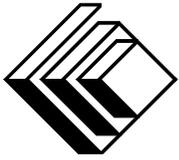
But we already know that the carry into the  $n$ th column is at most 22, since the carry from the  $n$ th column is the first time that the carry is at least 22.

This means that we have a contradiction, since the carry into the  $n$ th column cannot be both at most 22 and at least 34.

Thus, the only thing that can be wrong is our assumption that at some point the carry is at least 23.

Therefore, the carry is always at most 22.

This completes the argument, and shows that the total of the middle column is exactly 113, so the middle digit of the sum is 3.



**Canadian  
Mathematics  
Competition**

*An activity of the Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario*

***2008 Fryer Contest***

**Wednesday, April 16, 2008**

*Solutions*

1. (a) (i) The sum of the nine integers is

$$11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 = 135$$

- (ii) The nine numbers in the magic square add up to 135.

The sum of the numbers in each row is the same, so will be  $\frac{1}{3}$  of the total sum, or  $\frac{1}{3}(135) = 45$ .

Therefore, the magic constant is 45.

- (iii) The sum of the numbers in each row, in each column, and on each of the two main diagonals should be 45.

Since we are given two numbers in the first row and the top left to bottom right

diagonal, we can complete these to get 

18	11	16
	15	
		12

, since  $45 - 18 - 11 = 16$  and

$$45 - 18 - 12 = 15.$$

We can now complete the second and third columns to get 

18	11	16
	15	17
	19	12

.

We can now finish the magic square to get 

18	11	16
13	15	17
14	19	12

.

- (b) (i) The sum of the sixteen integers is

$$1 + 2 + 3 + 4 + \cdots + 13 + 14 + 15 + 16 = 136$$

(We can pair up the integers 1 with 16, 2 with 15, and so on, to get 8 pairs, each of which add to 17.)

- (ii) The sixteen numbers in the magic square add up to 136.

The sum of the numbers in each row is the same, so will be  $\frac{1}{4}$  of the total sum, or  $\frac{1}{4}(136) = 34$ .

Therefore, the magic constant is 34.

- (iii) The sum of the numbers in each row, in each column, and on each of the two main diagonals should be 34.

Since we are given three numbers in the first and fourth rows, first, third and fourth

columns, and on both diagonals, we can complete these to get 

16	3	2	13
5	10	11	8
9		7	12
4	15	14	1

,

since  $34 - 16 - 3 - 13 = 2$ , and so on.

We can now complete the magic square to get 

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

.

2. (a) After playing 5 more games, the Sharks had played  $10 + 5 = 15$  games in total and had won  $8 + 1 = 9$  of these.

Their final winning percentage was  $\frac{9}{15} \times 100\% = 60\%$ .

- (b) The Emus played  $10 + x$  games in total and won  $4 + x$  games in total.

Since their final winning percentage was 70%, then  $\frac{4+x}{10+x} \times 100\% = 70\%$  or  $\frac{4+x}{10+x} = \frac{7}{10}$ .

Therefore,  $10(4 + x) = 7(10 + x)$ , so  $40 + 10x = 70 + 7x$  or  $3x = 30$  or  $x = 10$ .

Therefore, the Emus played  $10 + x = 10 + 10 = 20$  games in total.

- (c) Suppose the Pink Devils played and lost  $y$  more games so that they had won exactly  $\frac{2}{7}$  of their games at that point.

Then they had played  $7 + 3 + y = 10 + y$  games in total and won 7 of them.

Therefore,  $\frac{7}{10 + y} = \frac{2}{7}$  or  $7(7) = 2(10 + y)$  or  $2y + 20 = 49$  or  $2y = 29$ .

But  $y$  is not an integer, so there was no point when the Pink Devils had won exactly  $\frac{2}{7}$  of their games.

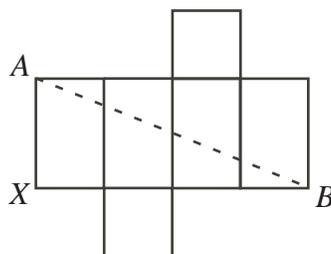
(We could have also argued starting with the equation  $7(7) = 2(10 + y)$  that the left side is even and the right side is odd, so there is no integer  $y$  that works.)

3. (a) The box that can be formed by folding the net in Figure 1 has dimensions 4 by 4 by 7, and so has volume  $4 \times 4 \times 7 = 112$ .

We can use the net more directly to calculate the surface area. The net shows that the box will have two faces that are 4 by 4, and four faces that are 4 by 7.

Therefore, the surface area is  $2(4 \times 4) + 4(4 \times 7) = 32 + 112 = 144$ .

- (b) We label point  $X$  on Figure 3.



Now  $AX$  is the height of the box, so  $AX = 6$ .

Also,  $\angle AXB = 90^\circ$ , since the box is rectangular.

Next, we see that  $XB = 2 + 2 + 2 + 2 = 8$ , since the four edges of the original box that form  $XB$  are those around the bottom face of the original box.

Therefore, by the Pythagorean Theorem,

$$AB^2 = AX^2 + XB^2 = 6^2 + 8^2 = 36 + 64 = 100$$

so  $AB = 10$ , since  $AB > 0$ .

- (c) We want to find the shortest path from  $A$  to  $G$  along the surface of the block. First, we make a few observations:

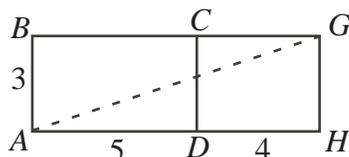
- Any path along the surface from  $A$  to  $G$  can be traced on an unfolded version of the block (that is, on a net like that in (a) or (b)).
- Given a particular way of unfolding the block, the shortest distance from  $A$  to  $G$  will be a straight line. The length of this straight line can be found using the Pythagorean Theorem, using the “rise” and the “run” from  $A$  to  $G$ , as in (b).
- In the given block, there is no single face that uses each of the vertices  $A$  and  $G$ . Thus, it is impossible for the caterpillar to travel from  $A$  to  $G$  along one face only, and so the caterpillar uses at least two faces.
- Any straight line path (and thus any path) that uses more than 2 faces will be longer than the possible straight line paths using exactly 2 faces. This is true because the rise and the run will be at least as long as in any of the 2 face paths below.

- Paths that include sections along edges do not need to be considered, as they will not form straight lines on the unfolded box.

So we need to examine all of the possible paths from  $A$  to  $G$  using exactly two faces. These combinations of faces are

- $ABCD$  then  $DCGH$
- $ABCD$  then  $BCGF$
- $ADHE$  then  $DCGH$
- $ADHE$  then  $EHGF$
- $ABFE$  then  $BCGF$
- $ABFE$  then  $EHGF$

Consider the straight line path across  $ABCD$  and  $DCGH$ .



The path is the hypotenuse of a right-angled triangle with legs of lengths 3 and 9, and so has length  $\sqrt{3^2 + 9^2} = \sqrt{90}$ .

In a similar way, the lengths of the paths are:

- $ABCD$  then  $DCGH$ :  $\sqrt{90}$
- $ABCD$  then  $BCGF$ :  $\sqrt{74}$
- $ADHE$  then  $DCGH$ :  $\sqrt{80}$
- $ADHE$  then  $EHGF$ :  $\sqrt{74}$
- $ABFE$  then  $BCGF$ :  $\sqrt{80}$
- $ABFE$  then  $EHGF$ :  $\sqrt{90}$

(We note that these lengths occur in three pairs of equal lengths, so we could have calculated three lengths and used symmetry to deal with the other three.)

Since the length of the shortest path is one of these lengths, then the shortest path has length  $\sqrt{74}$ .

4. (a) First, we examine the digits in a general palindrome.

If a palindrome  $P$  has an even number of digits (like 1221), then each digit from 0 to 9 will occur an even number of times in  $P$ , as each digit in the first half can be matched with a digit in the second half.

If a palindrome  $P$  has an odd number of digits (like 12321), then there will be one digit from 0 to 9 that occurs an odd number of times in  $P$  and every other digit will occur an even number of times. This is because  $P$  has a “middle” digit. If this digit is removed, the resulting number is a palindrome with an even number of digits and so has an even number of each of 0 to 9. Adding the “middle” digit back in, one of the digits occurs an odd number of times.

Any collection of digits, in which each occurs an even number of times, can be arranged to form a palindrome with an even number of digits by building the palindrome in pairs of digits from the centre out.

Similarly, any collection of digits in which each but one occurs an even number of times can be arranged to form a palindrome, by starting with one occurrence of the digit that

occurs the odd number of times in the middle, and building outwards.  
(No other collections of digits can be arranged to form a palindrome.)

Now, we examine  $x$ .

The integer  $x$  contains digits in the following distribution:

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	3	13	13	4	3	3	3	3	3	3

If we wanted to create a palindrome  $P$  with an even number of digits from the digits of  $x$ , we would need to reduce the number of occurrences of each of 0 to 9 so that each occurred an even number of times. To do this by removing the minimum number of digits, we can remove one each of 0, 1, 2, 4, 5, 6, 7, 8, 9. (To make an odd number even by subtracting the smallest amount, we subtract 1.) This ensures that each digit occurs an even number of times. Here, we have removed 9 digits in total.

If we wanted to create a palindrome  $P$  with an odd number of digits from the digits of  $x$ , we would need to reduce the number of occurrences of each of 0 to 9 so that all (but one) of them occurred an even number of times. To do this removing the minimum number of digits, we can remove 1 each of all but one of 0, 1, 2, 4, 5, 6, 7, 8, 9. (For the sake of argument, suppose that we do not remove a 9.) This ensures that each digit but 9 occurs an even number of times and 9 occurs an odd number of times. Here, we have removed 8 digits in total and formed a palindrome.

Therefore, the minimum number of digits that must be removed is 8.

- (b) From the chart above, the digits of  $x$  add to

$$3(0) + 13(1) + 13(2) + 4(3) + 3(5) + 3(6) + 3(7) + 3(8) + 3(9) = 168$$

To obtain a sum of 130, we must remove digits that have a sum of  $168 - 130 = 38$ .

We want to remove the minimum number of digits that have a sum of 38.

We try to do this by removing the largest digits first.

Removing three 9's, we have removed a total of 27.

Removing one 8, we have removed a total of 35.

We can now obtain a total removed of 38 by removing a 3.

Here, we have removed 5 digits.

We cannot remove 4 or fewer digits with a total of 38, as the maximum sum of 4 or fewer digits in theory is  $4(9) = 36$ .

Therefore, the minimum number of digits that we can remove is 5.

- (c) First, we enumerate the digits of  $y$ :

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	5	15	15	15	15	6	5	5	5	5

We can determine the sum of the digits by adding directly as above, or by noticing that each digit occurs at least 5 times, and grouping as follows:

$$5(0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) + 10(1 + 2 + 3 + 4) + 5 = 330$$

We want to remove digits from  $y$  whose sum is  $330 - 210 = 120$  and in such a way that, among the digits that remain, at most one of the digits 0 to 9 occurs an odd number of times.

We use the strategy of removing digits in such a way that each of 0 to 9 occurs an even number of times, and the digits sum to at most 210, and then perhaps add back in a single

digit (giving the final palindrome an odd number of digits) to make the sum equal to 210. We have to make each digit occur an even number of times, so first we remove 1 each of 0, 1, 2, 3, 4, 6, 7, 8, 9, to obtain:

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	4	14	14	14	14	6	4	4	4	4

whose sum is 290.

We now want to remove digits in pairs, removing the smallest *number* of digits that makes the total sum less than or equal to 210.

We do this by removing the largest possible digits first. Removing four 8's and four 9's gives

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	4	14	14	14	14	6	4	4	0	0

whose sum is 222.

We need to remove at least two more digits to obtain a sum of less than or equal to 210. (We cannot remove a single digit equal to at least 12.)

We could do this for example by removing two 6's, obtaining

Digit	0	1	2	3	4	5	6	7	8	9
Number of times	4	14	14	14	14	6	2	4	0	0

We could then add back in a single 0 to maintain the sum of 210.

In total, we have removed  $9 + 4 + 4 + 2 - 1 = 18$  digits.

This is the minimum number of digits because we have removed the minimum number of digits to form a palindrome with an even number of digits, and then added one digit back in.



**Canadian  
Mathematics  
Competition**

*An activity of the Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario*

***2007 Fryer Contest***

**Wednesday, April 18, 2007**

*Solutions*

1. (a) Since Rectangle 3 has 4 rows of squares, then Rectangle 4 has 5 rows.  
 Since Rectangle 3 has 7 columns of squares, then Rectangle 4 has 9 columns.  
 Therefore, Rectangle 4 has  $5 \times 9 = 45$  squares.
- (b) Since Rectangle 4 has 5 rows, its height is 5.  
 Since Rectangle 4 has 9 columns, its width is 9.  
 Therefore, the perimeter of Rectangle 4 is  $2(5) + 2(9) = 28$ .
- (c) Since Rectangle 4 has 5 rows of squares, then Rectangle 7 has  $5 + 3 = 8$  rows.  
 Since Rectangle 4 has 9 columns of squares, then Rectangle 7 has  $9 + 2(3) = 15$  columns.  
 Since Rectangle 7 has 8 rows and 15 columns, it is 8 by 15, so has perimeter  $2(8) + 2(15) = 46$ .
- (d) *Solution 1*  
 Rectangle 7 is 8 by 15 and so has perimeter  $2(8) + 2(15) = 46$ .  
 Let us try a bigger rectangle.  
 Rectangle 17 has  $8 + 10 = 18$  rows and  $15 + 2(10) = 35$  columns, so is 18 by 35 and has perimeter  $2(18) + 2(35) = 106$ .  
 Let us try a rectangle that is still bigger.  
 Rectangle 27 has  $18 + 10 = 28$  rows and  $35 + 2(10) = 55$  columns, so is 28 by 55 and has perimeter  $2(28) + 2(55) = 166$ .  
 We are getting close!  
 Rectangle 28 has 29 rows and 57 columns, so has perimeter  $2(29) + 2(57) = 172$ .  
 Rectangle 29 has 30 rows and 59 columns, so has perimeter  $2(30) + 2(59) = 178$ .  
 Therefore,  $n = 29$ .  
 (Notice that this is the only answer as moving more steps along in the sequence makes the rectangles larger.)

*Solution 2*

Rectangle 7 is 8 by 15 and so has perimeter  $2(8) + 2(15) = 46$ .  
 When we move to Rectangle 8, the height increases by 1 and the width increases by 2.  
 This increase the perimeter by  $2(1) + 2(2) = 6$ .  
 The same increase in perimeter occurs at each step in the sequence.  
 Since  $178 - 46 = 132 = 22(6)$ , then we must move 22 steps beyond Rectangle 7 to get from a perimeter of 46 to a perimeter of 178.  
 Thus, Rectangle 29 has a perimeter of 178, so  $n = 29$ .

*Solution 3*

Rectangle 1 has 1 more row than its number, and the number of rows increases by 1 from each step to the next.  
 Therefore, Rectangle  $n$  has  $n + 1$  rows.  
 Rectangle 1 has 3 columns (which is 1 more than twice its number), and the number of columns increases by 2 from each step to the next.  
 Therefore, Rectangle  $n$  has  $2n + 1$  columns.  
 A formula for the perimeter of Rectangle  $n$  is  $2(n+1) + 2(2n+1) = 2n+2+4n+2 = 6n+4$ .  
 For a perimeter of 178, we must have  $6n + 4 = 178$  or  $6n = 174$  or  $n = 29$ .  
 Therefore, Rectangle 29 has a perimeter of 178.

2. (a) Jim buys  $5 + 2 + 3 = 10$  tickets.  
 The total cost of the tickets is  $5(\$25) + 2(\$10) + 3(\$5) = \$160$ .

Therefore, the average cost of the tickets was  $\frac{\$160}{10} = \$16$ .

- (b) Since Mike buys 8 tickets at an average cost of \$12, their total cost is  $8 \times \$12 = \$96$ .

When he buys 5 more platinum tickets, he pays  $5 \times \$25 = \$125$ .

In total, Mike thus pays  $\$96 + \$125 = \$221$  for 13 tickets.

Therefore, the new average cost of the tickets is  $\frac{\$221}{13} = \$17$ .

- (c) *Solution 1*

The first 10 tickets that Ophelia buys at an average cost of \$14 have a total cost of  $10 \times \$14 = \$140$ .

When she buys  $n$  more platinum tickets, she pays  $25n$  dollars for these additional tickets.

In total, she has now paid  $140 + 25n$  dollars for  $10 + n$  tickets.

Since the average price of the tickets that she has bought is \$20, then

$$\begin{aligned}\frac{140 + 25n}{10 + n} &= 20 \\ 140 + 25n &= 20(10 + n) \\ 140 + 25n &= 200 + 20n \\ 5n &= 60 \\ n &= 12\end{aligned}$$

so she buys 12 more platinum tickets.

*Solution 2*

For the first ten tickets that Ophelia buys, the average cost is \$6 less per ticket than the final average cost of \$20.

Therefore, she has paid  $10 \times \$6 = \$60$  less in total than she would have if she had paid \$20 on average for these tickets.

For a final average of \$20 per ticket, the new platinum tickets must cost in total \$60 more than an average of \$20.

Since each platinum ticket costs \$5 more on average than the final average, then she buys  $\$60 \div \$5 = 12$  more platinum tickets.

3. (a) For 992 466 1A6 to be divisible by 8, we must have 1A6 divisible by 8.

We check each of the possibilities, using a calculator or by checking by hand:

106 is not divisible by 8, 116 is not divisible by 8, 126 is not divisible by 8,

136 is divisible by 8,

146 is not divisible by 8, 156 is not divisible by 8, 166 is not divisible by 8,

176 is divisible by 8,

186 is not divisible by 8, 196 is not divisible by 8

Therefore, the possible values of  $A$  are 3 and 7.

- (b) For  $D767E89$  to be divisible by 9, we must have  $D + 7 + 6 + 7 + E + 8 + 9 = 37 + D + E$  divisible by 9.

Since  $D$  and  $E$  are each a single digit then each is between 0 and 9, so  $D + E$  is between 0 and 18.

Therefore,  $37 + D + E$  is between 37 and 55.

The numbers between 37 and 55 that are divisible by 9 are 45 and 54.

If  $37 + D + E = 45$ , then  $D + E = 8$ .

If  $37 + D + E = 54$ , then  $D + E = 17$ .

Therefore, the possible values of  $D + E$  are 8 and 17.

- (c) For  $541G5072H6$  to be divisible by 72, it must be divisible by 8 and by 9.

It is easier to check for divisibility by 8 first, since this will allow us to determine a small number of possibilities for  $H$ .

For  $541G5072H6$  to be divisible by 8, we must have  $2H6$  divisible by 8.

Going through the possibilities as in part (a), we can find that  $2H6$  is divisible by 8 when  $H = 1, 5, 9$  (that is, 216, 256 and 296 are divisible by 8 while 206, 226, 236, 246, 266, 276, 286 are not divisible by 8).

We must now use each possible value of  $H$  to find the possible values of  $G$  that make  $541G5072H6$  divisible by 9.

First,  $H = 1$ . What value(s) of  $G$  make  $541G507216$  divisible by 9?

In this case, we need  $5 + 4 + 1 + G + 5 + 0 + 7 + 2 + 1 + 6 = 31 + G$  divisible by 9.

Since  $G$  is between 0 and 9, then  $31 + G$  is between 31 and 40, so must be equal to 36 if it is divisible by 9. Thus,  $G = 5$ .

Next,  $H = 5$ . What value(s) of  $G$  make  $541G507256$  divisible by 9?

In this case, we need  $5 + 4 + 1 + G + 5 + 0 + 7 + 2 + 5 + 6 = 35 + G$  divisible by 9.

Since  $G$  is between 0 and 9, then  $35 + G$  is between 35 and 44, so must be equal to 36 if it is divisible by 9. Thus,  $G = 1$ .

Last,  $H = 9$ . What value(s) of  $G$  make  $541G507296$  divisible by 9?

In this case, we need  $5 + 4 + 1 + G + 5 + 0 + 7 + 2 + 9 + 6 = 39 + G$  divisible by 9.

Since  $G$  is between 0 and 9, then  $39 + G$  is between 39 and 48, so must be equal to 45 if it is divisible by 9. Thus,  $G = 6$ .

Therefore, the possible pairs of values are  $H = 1$  and  $G = 5$ ,  $H = 5$  and  $G = 1$ , and  $H = 9$  and  $G = 6$ .

(Note that we could have combined the analysis of these last three cases.)

4. (a) *Solution 1*

By the Pythagorean Theorem,  $YZ^2 = YX^2 + XZ^2 = 60^2 + 80^2 = 3600 + 6400 = 10000$ , so  $YZ = 100$ .

(We could also have found  $YZ$  without using the Pythagorean Theorem by noticing that  $\triangle XYZ$  is a right-angled triangle with its right-angle at  $X$  and  $XY = 60 = 3(20)$  and  $XZ = 80 = 4(20)$ . This means that  $\triangle XYZ$  is similar to a 3-4-5 triangle, so has  $YZ = 5(20) = 100$ .)

Since  $\triangle YXZ$  is right-angled at  $X$ , its area is  $\frac{1}{2}(60)(80) = 2400$ .

Since  $XW$  is perpendicular to  $YZ$ , then the area of  $\triangle YXZ$  is also equal to  $\frac{1}{2}(100)(XW) = 50XW$ .

Therefore,  $50XW = 2400$ , so  $XW = 48$ .

By the Pythagorean Theorem,  $WZ^2 = 80^2 - 48^2 = 6400 - 2304 = 4096$ .

Thus,  $WZ = \sqrt{4096} = 64$ .

*Solution 2*

By the Pythagorean Theorem,  $YZ^2 = YX^2 + XZ^2 = 60^2 + 80^2 = 3600 + 6400 = 10000$ , so  $YZ = 100$ .

Let  $WZ = a$ . Then  $YW = 100 - a$ .

Let  $XW = h$ .

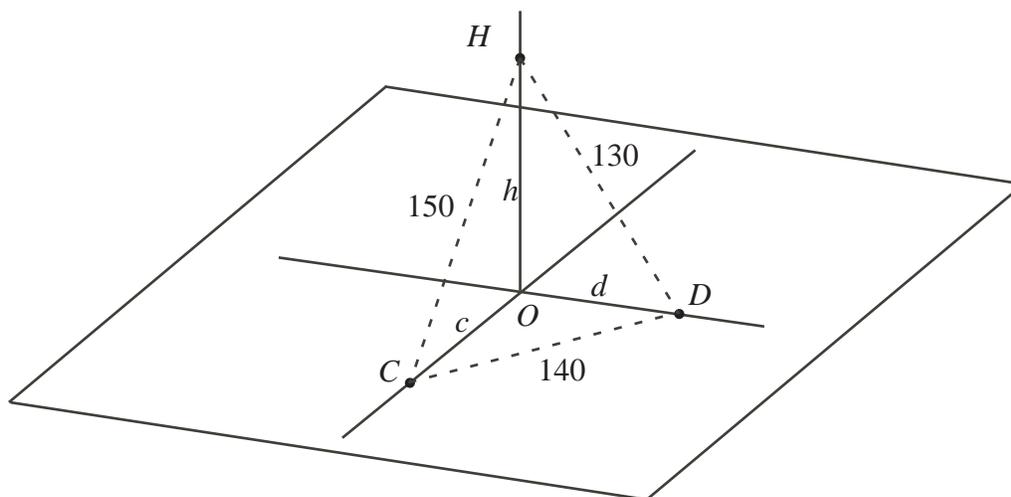
By the Pythagorean Theorem in  $\triangle XWY$ , we have  $(100 - a)^2 + h^2 = 60^2$ .

By the Pythagorean Theorem in  $\triangle XWZ$ , we have  $a^2 + h^2 = 80^2$ .  
Subtracting the first of these equations from the second, we obtain

$$\begin{aligned} a^2 - (100 - a)^2 &= 80^2 - 60^2 \\ a^2 - (10000 - 200a + a^2) &= 6400 - 3600 \\ 200a - 10000 &= 2800 \\ 200a &= 12800 \\ a &= 64 \end{aligned}$$

Therefore,  $WZ = 64$ .

(b) Let  $OC = c$ ,  $OD = d$  and  $OH = h$ .

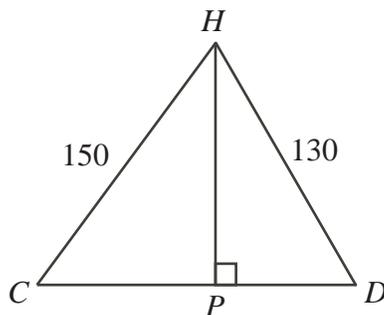


Note that  $OH$  is perpendicular to the field, so  $OH$  is perpendicular to  $OC$  and to  $OD$ .  
Also, since  $OD$  points east and  $OC$  points south, then  $OD$  is perpendicular to  $OC$ .  
Since  $HC = 150$ , then  $h^2 + c^2 = 150^2$  by the Pythagorean Theorem.  
Since  $HD = 130$ , then  $h^2 + d^2 = 130^2$ .  
Since  $CD = 140$ , then  $c^2 + d^2 = 140^2$ .  
Adding the first two equations, we obtain  $2h^2 + c^2 + d^2 = 150^2 + 130^2$ .  
Since  $c^2 + d^2 = 140^2$ , then

$$\begin{aligned} 2h^2 + 140^2 &= 150^2 + 130^2 \\ 2h^2 &= 150^2 + 130^2 - 140^2 \\ 2h^2 &= 19800 \\ h^2 &= 9900 \\ h &= \sqrt{9900} = 30\sqrt{11} \end{aligned}$$

Therefore, the height of the balloon above the field is  $30\sqrt{11} \approx 99.5$  m.

(c) To save the most rope, we must have  $HP$  having minimum length.  
For  $HP$  to have minimum length,  $HP$  must be perpendicular to  $CD$ .



(Among other things, we can see from this diagram that sliding  $P$  away from the perpendicular position does make  $HP$  longer.)

In the diagram,  $HC = 150$ ,  $HD = 130$  and  $CD = 140$ .

Let  $HP = x$  and  $PD = a$ . Then  $CP = 140 - a$ .

By the Pythagorean Theorem in  $\triangle HPC$ ,  $x^2 + (140 - a)^2 = 150^2$ .

By the Pythagorean Theorem in  $\triangle HPD$ ,  $x^2 + a^2 = 130^2$ .

Subtracting the second equation from the first, we obtain

$$\begin{aligned} (140 - a)^2 - a^2 &= 150^2 - 130^2 \\ (19600 - 280a + a^2) - a^2 &= 5600 \\ 19600 - 280a &= 5600 \\ 280a &= 14000 \\ a &= 50 \end{aligned}$$

Therefore,  $x^2 + 90^2 = 150^2$  or  $x^2 = 150^2 - 90^2 = 22500 - 8100 = 14400$  so  $x = 120$ .

So the shortest possible rope that we can use is 120 m, which saves  $130 + 150 - 120 = 160$  m of rope.



**Canadian  
Mathematics  
Competition**

*An activity of the Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario*

***2006 Fryer Contest***

**Thursday, April 20, 2006**

*Solutions*

1. (a) Her average mark in the seven courses is

$$\frac{94 + 93 + 84 + 81 + 74 + 83 + 79}{7} = \frac{588}{7} = 84$$

- (b) Samantha's highest possible average would be if she obtained a mark of 100 in French. In this case, her average in the eight courses would be

$$\frac{94 + 93 + 84 + 81 + 74 + 83 + 79 + 100}{8} = \frac{688}{8} = 86$$

- (c) *Solution 1*

If Samantha's final average over all eight courses is 85, then the sum of her marks in the eight courses is  $8 \times 85 = 680$ .

In part (a), we saw that the sum of her first seven marks is 588, so her mark in French is  $680 - 588 = 92$ .

*Solution 2*

Samantha's average in the first seven courses is 84, so if her French mark was 84, then her average would remain as 84.

If Samantha's mark in French was 100 (as in (b)), her average would be 86.

Since her average is 85 (half-way between these two averages), then her French mark is half-way between 84 and 100, or 92.

2. (a) The bottom layer of the cube is a 7 by 7 square of cubes, so uses  $7 \times 7 = 49$  cubes. The next layer of the cube is a 5 by 5 square of cubes, so uses  $5 \times 5 = 25$  cubes. The next layer of the cube is a 3 by 3 square of cubes, so uses  $3 \times 3 = 9$  cubes. The top layer consists of a single cube. Therefore, the total number of cubes used is  $49 + 25 + 9 + 1 = 84$ .

- (b) *Solution 1*

The cube in the top layer has 5 visible faces (only the bottom face is hidden).

In the second layer from the top, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there is 1 cube on each of the four sides of the layer with 2 visible faces (another 8 visible faces).

In the second layer from the bottom, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there are 3 cubes on each of the four sides of the layer with 2 visible faces (another  $4 \times 3 \times 2 = 24$  visible faces).

In the bottom layer, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there are 5 cubes on each of the four sides of the layer with 2 visible faces (another  $4 \times 5 \times 2 = 40$  visible faces).

Therefore, the total number of visible faces is  $5 + 12 + 8 + 12 + 24 + 12 + 40 = 113$ .

*Solution 2*

When we look at the pyramid from the top, we see a  $7 \times 7$  square of visible faces, or 49 visible faces. (This square is composed of faces from all of the levels of the pyramid.)

When we look at the pyramid from each of the four sides, we see  $1 + 3 + 5 + 7 = 16$  visible faces, so there are  $4(16) = 64$  visible faces on the sides.

Therefore, there are  $49 + 64 = 113$  visible faces in total.

(c) *Solution 1*

To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three or five numbers are visible, depending on its position.

For the top cube (with 5 visible faces), we position this cube with the “1” on the bottom face (and so is hidden).

The total of the numbers visible on this cube is  $2 + 3 + 4 + 5 + 6 = 20$ .

For the 4 corner cubes on each layer (each with 3 visible faces), we position these cubes with the 4, 5 and 6 all visible (this is possible since the faces with the 4, 5 and 6 share a vertex) and the 1, 2 and 3 hidden.

There are 12 of these cubes, so the total of the numbers visible on these cubes is  $12 \times (4 + 5 + 6) = 180$ .

For the cubes on the sides (that is, not at the corner) of each layer, we position the cubes with the 5 and 6 visible (this is possible since the faces with 5 and 6 share an edge) and the 1, 2, 3, and 4 hidden.

There are  $4 + 12 + 20 = 36$  of these cubes, so the total of the numbers visible on these cubes is  $36 \times (5 + 6) = 396$ .

Therefore, the overall largest possible total is  $20 + 180 + 396 = 596$ .

*Solution 2*

To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three or five numbers are visible, depending on its position.

As in Solution 2 to (b), view the pyramid from the top. Position the cubes so that each top faces which is visible is 6 (for a total of  $49 \times 6 = 294$ ).

Consider next the top cube. It has only one hidden face, which will be the 1, since the 6 is on top. (This does maximize the sum of the visible faces.) This adds  $2 + 3 + 4 + 5 = 14$  to the total of the visible faces thus far.

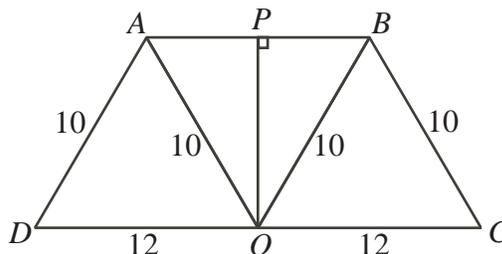
Consider lastly the faces visible on the sides (a total of  $4(3 + 5 + 7) = 60$  faces). To maximize the sum of the numbers on these faces, we would like to make them all 5 (since we have already used the 6s). This would give a total of  $5 \times 60 = 300$ . However, there are 12 corner cubes to which we have assigned two 5s, so we must change one of the 5s on each to a 4, decreasing the total by 12. (This is possible, since the 4, 5 and 6 meet at a vertex on each cube.)

Therefore, the overall largest possible total is  $294 + 14 + 300 - 12 = 596$ .

3. (a) Since  $\triangle AOB$  is isosceles with  $AO = OB$  and  $OP$  is perpendicular to  $AB$ , then  $P$  is the midpoint of  $AB$ , so  $AP = PB = \frac{1}{2}AB = \frac{1}{2}(12) = 6$ .  
By the Pythagorean Theorem,  $OP = \sqrt{AO^2 - AP^2} = \sqrt{10^2 - 6^2} = \sqrt{64} = 8$ .

(b) *Solution 1*

Trapezoid  $ABCD$  is formed from three congruent triangles, so its area is three times the area of one of these triangles.



Each triangle has a base of length 12 and a height of length 8 (from (a), since  $OP$  is one of these heights), so has area  $\frac{1}{2}(12)(8) = 48$ .  
Therefore, the area of the trapezoid is  $3 \times 48 = 144$ .

*Solution 2*

Since  $ABCD$  is a trapezoid with height of length 8 ( $OP$  is the height of  $ABCD$ ) and parallel sides ( $AB$  and  $DC$ ) of length 12 and 24, then its area is

$$\frac{1}{2} \times \text{Height} \times \text{Sum of parallel sides} = \frac{1}{2}(8)(12 + 24) = 144$$

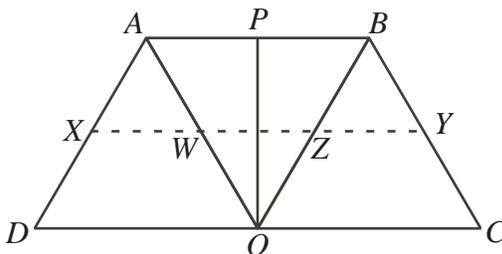
(c) *Solution 1*

Since  $XY$  cuts  $AD$  and  $BC$  each in half, then it also cuts the height  $PO$  in half:

Since  $XY$  is parallel to  $AB$  and  $DC$  and cuts each of  $AD$  and  $BC$  at its midpoint, then it must cut each of  $AO$  and  $BO$  at its midpoint ( $W$  and  $Z$ , respectively).  
Therefore,  $\triangle WZO$  is similar to  $\triangle ABO$  and its side lengths are half of those of  $\triangle ABO$ .

Thus, its height is half of  $PO$ .

In a similar way, the dimensions of  $\triangle AXW$  and  $\triangle BYZ$  are half of those of  $\triangle ADO$  and  $\triangle BCO$ .



Thus, each of the two smaller trapezoids has height 4.

Also, since  $XW = \frac{1}{2}DO$ ,  $WZ = \frac{1}{2}AB$  and  $ZY = \frac{1}{2}OC$ , then  $XY = 3(6) = 18$ .

Using the formula for the area of a trapezoid from (b), the area of trapezoid  $ABYX$  is  $\frac{1}{2}(4)(12 + 18) = 60$  and the area of trapezoid  $XYCD$  is  $\frac{1}{2}(4)(18 + 24) = 84$ .

Thus, the ratio of their areas is  $60 : 84 = 5 : 7$ .

*Solution 2*

Since  $XY$  cuts  $AD$  and  $BC$  each in half, then it also cuts the height  $PO$  in half.

Thus, each of the two smaller trapezoids has height 4.

Next, we find the length of  $XY$ .

From (b), we know how to compute the area of a trapezoid and we know that the sum of the areas of trapezoids  $ABYX$  and  $XYCD$  must equal that of trapezoid  $ABCD$ .

Therefore,

$$\begin{aligned}\frac{1}{2}(4)(AB + XY) + \frac{1}{2}(4)(XY + DC) &= 144 \\ 2(12 + XY) + 2(XY + 24) &= 144 \\ 4(XY) &= 72 \\ XY &= 18\end{aligned}$$

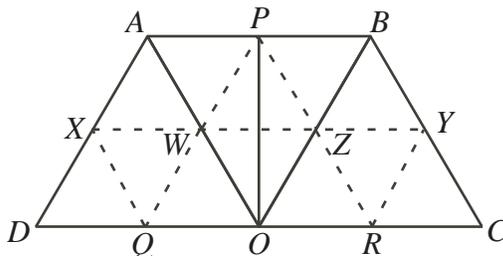
Thus, the area of trapezoid  $ABYX$  is  $\frac{1}{2}(4)(12+18) = 60$  and the area of trapezoid  $XYCD$  is  $\frac{1}{2}(4)(18 + 24) = 84$ .

Thus, the ratio of their areas is  $60 : 84 = 5 : 7$ .

*Solution 3*

Let  $Q$  and  $R$  be the midpoints of  $DO$  and  $OC$ , respectively, and  $W$  and  $Z$  the points where  $XY$  crosses  $AO$  and  $BO$ , respectively.

Join  $X$  and  $W$  to  $Q$ ,  $Z$  and  $Y$  to  $R$ , and  $W$  and  $Z$  to  $P$ .



We have now divided trapezoid  $ABCD$  into 12 congruent triangles:

To show this, we consider the partitioning of  $\triangle AOD$  into four small triangles.

Because  $X$ ,  $W$  and  $Q$  are the midpoints of  $AD$ ,  $AO$  and  $OD$ , respectively, then  $XW$ ,  $WQ$  and  $QX$  are parallel to  $DO$ ,  $AD$  and  $OA$ , respectively.

Therefore, triangles  $AXW$ ,  $XDQ$ ,  $WQO$ , and  $QWX$  are congruent, because each has sides of length 5, 5 and 6.

The same is true for the partitioning of  $\triangle ABO$  and  $\triangle BOC$ .

Since  $ABYX$  is made up of 5 of these triangles and  $XYCD$  is made up of 7 of these triangles and each of these 12 triangles has equal area, then the ratio of the area of  $ABYX$  to the area of  $XYCD$  is  $5 : 7$ .

4. (a) *Solution 1*

There are 100 integers from 1 to 100.

Of these, 10 end with the digit 7: 7, 17, 27, 37, 47, 57, 67, 77, 87, 97.

Also, 10 begin with the digit 7: 70, 71, 72, 73, 74, 75, 76, 77, 78, 79.

But 77 is included in both lists, so we need to be careful not to count it twice. Therefore, we need to subtract  $10 + 10 - 1 = 19$  of the 100 integers.

Therefore, the number of these integers which do not contain 7 is  $100 - 19 = 81$ .

*Solution 2*

The integer 100 does not contain the digit 7, nor does the integer 0, so we replace 100 with 0 in the list and count the number of integers from 0 to 99 which do not contain the

digit 7.

Each of these integers can be written as a two-digit integer (where we allow the first digit to be 0): 00, 01, 02, ..., 98, 99.

Since we would like all integers not containing the digit 7, there are 9 possibilities for the first digit (0 through 9, excluding 7) and for each of these possibilities, there are 9 possibilities for the second digit.

Therefore, there are  $9 \times 9 = 81$  integers in this range which do not contain the digit 7.

(b) *Solution 1*

From part (a), there are 81 integers not containing the digit 7 between 1 and 100.

Similarly, there will be 81 such integers in each of the ranges 101 to 200, 201 to 300, 301 to 400, 401 to 500, and 501 to 600. (We can conclude this since the first digit is not a 7, and the tens and units digits follow the same pattern as in (a).)

From 601 to 700, there will be 80 such integers (since 700 includes 7).

From 701 to 800, there is only 1 such integer (namely, 800 – all others contain a 7).

In each of the intervals 801 to 900, 901 to 1000, 1001 to 1100, 1101 to 1200, 1201 to 1300, 1301 to 1400, 1401 to 1500, and 1501 to 1600, there are 81 such integers.

From 1601 to 1700, there are 80; from 1701 to 1800, there is 1; from 1801 to 1900 and 1901 to 2000, there are 81.

Therefore, the total number of such integers is  $16(81) + 2(80) + 2(1) = 18(81) = 1458$ .

*Solution 2*

The integer 2000 does not contain the digit 7, nor does the integer 0, so we can replace 2000 with 0 in the list and count the number of integers from 0 to 1999 which do not contain the digit 7.

Each of these integers can be written as a four-digit integer  $\underline{a}\underline{b}\underline{c}\underline{d}$ , where the integer is allowed to begin with one or more zeros, with  $a$  allowed to be 0 or 1, and each of  $b$ ,  $c$  and  $d$  allowed to be 0, 1, 2, 3, 4, 5, 6, 8, or 9.

Thus, there are 2 possibilities for the first digit; for each of these choices, there are 9 choices for the second digit; for each of these choices, there are 9 choices for the third digit; for each of these choices, there are 9 choices for the fourth digit. The total number of such integers is thus  $2 \times 9 \times 9 \times 9 = 1458$ .

(c) *Solution 1*

In this solution, we will repeatedly use the fact that the sum of the integers from 1 to  $n$  is  $\frac{1}{2}n(n+1)$ .

Consider first the integers from 1 to 100.

The sum of these integers is  $\frac{1}{2}(100)(101) = 5050$ .

The integers in this set which do contain the digit 7 are 7, 17, 27, 37, 47, 57, 67, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 87, 97, whose sum is 1188.

Therefore, the sum of the integers from 1 to 100 which do not contain the digit 7 is  $5050 - 1188 = 3862$ .

There are 81 numbers from 101 to 200 not containing the digit 7 as well. Each of these is 100 more than a corresponding number between 1 and 100 which does not contain the digit 7, so the sum of these 81 numbers is  $3862 + 81(100)$ .

We can use this approach to determine the sum of the appropriate numbers in each range of 100, as shown in the table:

Range	Number of integers not containing 7	Sum
1 to 100	81	3862
101 to 200	81	3862 + 81(100)
201 to 300	81	3862 + 81(200)
301 to 400	81	3862 + 81(300)
401 to 500	81	3862 + 81(400)
501 to 600	81	3862 + 81(500)
601 to 700	80	3862 + 81(600) - 700
701 to 800	1	800
801 to 900	81	3862 + 81(800)
901 to 1000	81	3862 + 81(900)
1001 to 1100	81	3862 + 81(1000)
1101 to 1200	81	3862 + 81(1100)
1201 to 1300	81	3862 + 81(1200)
1301 to 1400	81	3862 + 81(1300)
1401 to 1500	81	3862 + 81(1400)
1501 to 1600	81	3862 + 81(1500)
1601 to 1700	80	3862 + 81(1600) - 1700
1701 to 1800	1	1800
1801 to 1900	81	3862 + 81(1800)
1901 to 2000	81	3862 + 81(1900)
2001 to 2006	6	2001 + 2002 + 2003 + 2004 + 2005 + 2006

Therefore, the overall sum is

$$\begin{aligned}
& 18(3862) \\
& \quad + 81(100)(1 + 2 + 3 + 4 + 5 + 6 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 18 + 19) \\
& \quad - 700 + 800 - 1700 + 1800 + (2001 + 2002 + 2003 + 2004 + 2005 + 2006) \\
& = 69516 + 8100(166) + 200 + 12021 \\
& = 1426337
\end{aligned}$$

*Solution 2*

Consider first the integers from 000 to 999 that do not contain the digit 7. (We can include 000 in this list as it will not affect the sum.)

Since each of the three digits has 9 possible values, there are  $9 \times 9 \times 9 = 729$  such integers. If we fix any specific digit in any of the three positions, there will be exactly 81 integers with that digit in that position, as there are 9 possibilities for each of the remaining digits. (For example, there are 81 such integers ending in 0, 81 ending in 1, etc.)

We sum these integers by first summing the units digits, then summing the tens digits, and then summing the hundreds digits.

Since each of the 9 possible units digits occurs 81 times, the sum of the units digits column is

$$81(0) + 81(1) + 81(2) + 81(3) + 81(4) + 81(5) + 81(6) + 81(8) + 81(9) = 81(38)$$

Since each of the 9 possible tens digits occurs 81 times, the sum of the tens digits column is  $81(0 + 10 + 20 + 30 + 40 + 50 + 60 + 80 + 90) = 81(380)$ .

Similarly, the sum of the hundreds digits column is  $81(3800)$ .

Thus, the sum of the integers from 0 to 999 that do not contain the digit 7 is

$$81(38) + 81(380) + 81(3800) = 81(38)(1 + 10 + 100) = 81(38)(111) = 341\,658$$

Each of the 729 integers from 1000 to 1999 which do not contain 7 is 1000 more than such an integer between 0 and 999. (There are again 729 of these integers as the first digit is fixed at 1, and each of the remaining three digits has 9 possible values.)

Thus, the sum of these integers from 1000 to 1999 is equal to the sum of the corresponding ones from 0 to 999 plus  $729(1000)$ , or  $729\,000 + 341\,658 = 1\,070\,658$ .

Each of the integers from 2000 to 2006 does not contain a 7, and their sum is

$$2000 + 2001 + 2002 + 2003 + 2004 + 2005 + 2006 = 7(2003) = 14\,021.$$

Therefore, the overall sum is  $341\,658 + 1\,070\,658 + 14\,021 = 1\,426\,337$ .



**Canadian  
Mathematics  
Competition**

*An activity of the Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario*

***2005 Fryer Contest***

***Wednesday, April 20, 2005***

*Solutions*

1. (a) The area of a circle with radius  $r$  is  $\pi r^2$ .  
 So the area of the larger circle is  $\pi(10^2) = 100\pi$  and the area of the smaller circle is  $\pi(6^2) = 36\pi$ .  
 The area of the ring between the two circles is the difference of these two areas.  
 Therefore, the area of the ring is  $100\pi - 36\pi = 64\pi$ .
- (b) The area of the inside circle (region  $X$ ) is  $\pi(4^2) = 16\pi$ .  
 Using a similar technique to part (a), the area of the middle ring (region  $Y$ ) is  $\pi(6^2) - \pi(4^2) = 36\pi - 16\pi = 20\pi$ .  
 Also, the area of the outer ring (region  $Z$ ) is  $\pi(7^2) - \pi(6^2) = 49\pi - 36\pi = 13\pi$ .  
 Therefore, region  $Y$  has the largest area.
- (c) The area of the ring between the two largest circles is  $\pi(13^2) - \pi(12^2) = 169\pi - 144\pi = 25\pi$ .  
 Suppose that the radius of the smallest circle is  $r$ .  
 Thus, the area of the smallest circle is  $\pi r^2$ .  
 Since the area of the smallest circle is equal to the area of the ring between the two largest circles, then  $\pi r^2 = 25\pi$  so  $r^2 = 25$  and so  $r = 5$  since  $r > 0$ .  
 Therefore, the radius of the smallest circle is 5.
2. (a) Anh goes first and puts an A in the middle box.  
 According to the rules, Bharati must put her initial in one or two boxes which are next to each other. So Bharati can only put a B in one of the two empty boxes, since they are not next to each other.



This leaves one empty box, and Anh wins by putting an A in this box.  
 Therefore, Anh will win no matter what Bharati does.

(b) *Solution 1*

Suppose Anh puts an A in the rightmost box.



Then Bharati can only put a B in one of the two empty boxes, since they are not next to each other.

This leaves one empty box, and Anh will win by putting an A in this empty box.  
 Therefore, Anh is guaranteed to win if he puts an A in the rightmost box.

*Solution 2*

Suppose Anh puts an A in the second box from the right end.



Then Bharati can only put a B in one of the two empty boxes, since they are not next to each other.

This leaves one empty box, and Anh will win by putting an A in this empty box.  
 Therefore, Anh is guaranteed to win if he puts an A in the second box from the right.

*Solution 3*

Suppose Anh puts an A in the rightmost box.

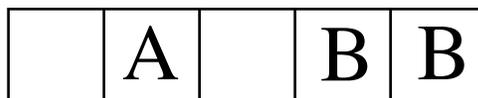


We can now remove the leftmost and rightmost boxes, leaving us with three boxes with an A in the middle and it being Bharati's turn.

This leaves us in the situation of part (a), so Anh will be guaranteed to win.

(c) Possibility #1

Suppose Bharati puts a B in the two rightmost boxes.



Then Anh can only put an A in one of the two empty boxes, since they are not next to each other.

This leaves one empty box, and Bharati wins by putting a B in this empty box.

Therefore, Bharati is guaranteed to win in this case.

Possibility #2

Suppose Bharati puts a B in the middle box and the box to its right.



Then Anh can only put an A in one of the two empty boxes, since they are not next to each other.

This leaves one empty box, and Bharati wins by putting a B in this empty box.

Therefore, Bharati is guaranteed to win in this case.

These are the two possible moves that Bharati can make next to guarantee she wins.

(The only other four possible moves are putting a B in any one of the empty boxes. Why do each of these four moves allow A to win?)

3. (a) *Solution 1*

Since the side lengths of a Nakamoto triangle are in the ratio 3 : 4 : 5, then the side lengths must be the products 3, 4 and 5 with the same integer.

For one of the sides to have a length of 28, it must be the multiple of 4, since 28 is not a multiple of 3 or 5.

Since  $28 = 4 \times 7$ , then the three side lengths must be  $3 \times 7 = 21$ , 28 and  $5 \times 7 = 35$ .

*Solution 2*

Since the side lengths of a Nakamoto triangle are three integers in the ratio 3 : 4 : 5, then the side lengths are  $3x$ ,  $4x$  and  $5x$ , for some positive integer  $x$ .

Since one of these three sides is 28, then we must have  $4x = 28$  (or  $x = 7$ ), because 28 is not a multiple of 3 or 5.

Therefore, the side lengths are  $3 \times 7 = 21$ , 28 and  $5 \times 7 = 35$ .

(b) *Solution 1*

Since the side lengths of a Nakamoto triangle are in the ratio  $3 : 4 : 5$ , then the side lengths must be the products  $3$ ,  $4$  and  $5$  with the same integer.

The Nakamoto triangle with the shortest sides is that with side lengths  $3$ ,  $4$  and  $5$ , which has a perimeter of  $3 + 4 + 5 = 12$ .

Since we are given the Nakamoto triangle of perimeter  $96$  and  $96 = 12 \times 8$ , then we must multiply each of the side lengths of the triangle with sides  $3$ ,  $4$  and  $5$  by  $8$  to get a perimeter of  $96$ .

Therefore, the side lengths are  $3 \times 8 = 24$ ,  $4 \times 8 = 32$  and  $5 \times 8 = 40$ .

*Solution 2*

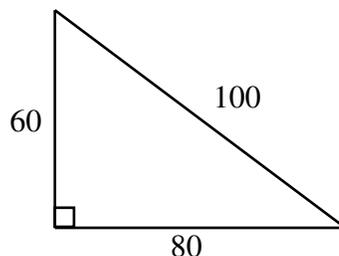
Since the side lengths of a Nakamoto triangle are three integers in the ratio  $3 : 4 : 5$ , then the side lengths are  $3x$ ,  $4x$  and  $5x$ , for some integer  $x$ .

Since the perimeter is  $96$ , then  $3x + 4x + 5x = 96$  or  $12x = 96$  or  $x = 8$ .

Therefore, the side lengths are  $3 \times 8 = 24$ ,  $4 \times 8 = 32$  and  $5 \times 8 = 40$ .

- (c) Since  $60$  is divisible by  $3$ , then there is a Nakamoto triangle with a side length of  $60$  in the “ $3$ ” position. Since  $60 = 3 \times 20$ , then the side lengths of this triangle are  $60$ ,  $4 \times 20 = 80$  and  $5 \times 20 = 100$ .

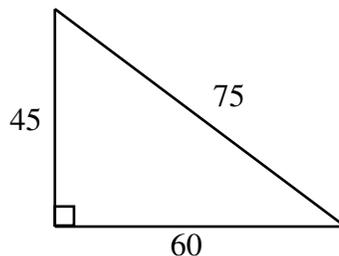
Since the ratio of the side lengths are  $3 : 4 : 5$ , then this triangle is right-angled. In fact, since  $60^2 + 80^2 = 100^2$ , then by the Pythagorean Theorem, the right angle is between the sides of lengths  $60$  and  $80$ . (Alternatively, we could have said that since the triangle is right-angled and  $100$  is the longest side, then  $100$  must be the hypotenuse, so the right angle is between the sides of lengths  $60$  and  $80$ .)



Therefore, the area of this triangle is  $\frac{1}{2}bh = \frac{1}{2}(60)(80) = 2400$ .

Since  $60$  is divisible by  $4$ , then there is a Nakamoto triangle with a side length of  $60$  in the “ $4$ ” position. Since  $60 = 4 \times 15$ , then the side lengths of this triangle are  $3 \times 15 = 45$ ,  $60$  and  $5 \times 15 = 75$ .

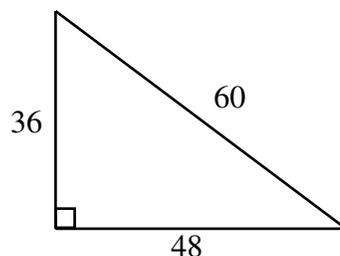
Since the triangle is right-angled and  $75$  is the longest side, then  $75$  must be the hypotenuse, so the right angle is between the sides of lengths  $45$  and  $60$ .



Therefore, the area of this triangle is  $\frac{1}{2}bh = \frac{1}{2}(45)(60) = 1350$ .

Since 60 is divisible by 5, then there is a Nakamoto triangle with a side length of 60 in the “5” position. Since  $60 = 5 \times 12$ , then the side lengths of this triangle are  $3 \times 12 = 36$ ,  $4 \times 12 = 48$  and 60.

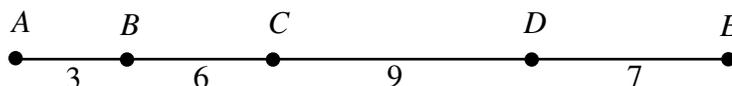
Since the triangle is right-angled and 60 is the longest side, then 60 must be the hypotenuse, so the right angle is between the sides of lengths 36 and 48.



Therefore, the area of this triangle is  $\frac{1}{2}bh = \frac{1}{2}(36)(48) = 864$ .

Thus, the possible areas of a Nakamoto triangle with a side length of 60 are 2400, 1350 and 864.

4. (a)



Here,  $AB = 3$ ,  $AC = 9$ ,  $AD = 18$ ,  $AE = 25$ ,  $BC = 6$ ,  $BD = 15$ ,  $BE = 22$ ,  $CD = 9$ ,  $CE = 16$ ,  $DE = 7$ .

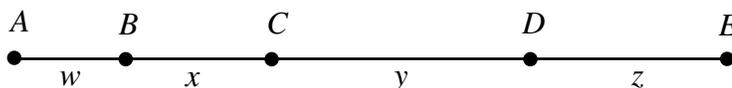
Therefore, the super-sum of  $AE$  is  $3 + 9 + 18 + 25 + 6 + 15 + 22 + 9 + 16 + 7 = 130$ .

(b) *Solution 1*

If the sub-segments have lengths 1 to 10, then the super-sum is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55.$$

Suppose that the basic sub-segments have lengths  $AB = w$ ,  $BC = x$ ,  $CD = y$  and  $DE = z$ .



Since all of the sub-segments have integer lengths, then all of the basic sub-segments must have integer length.

What are the lengths of the sub-segments in terms of  $w$ ,  $x$ ,  $y$  and  $z$ ?

We have  $AB = w$ ,  $AC = w + x$ ,  $AD = w + x + y$ ,  $AE = w + x + y + z$ ,  $BC = x$ ,  $BD = x + y$ ,  $BE = x + y + z$ ,  $CD = y$ ,  $CE = y + z$ , and  $DE = z$ .

Then the super-sum of  $AE$  is

$$w + (w+x) + (w+x+y) + (w+x+y+z) + x + (x+y) + (x+y+z) + y + (y+z) + z = 4w + 6x + 6y + 4z$$

Therefore, the super-sum is always an even number since

$$4w + 6x + 6y + 4z = 2(2w + 3x + 3y + 2z).$$

This tells us that the super-sum cannot be 55.

Thus, it is impossible for the line segment  $AE$  to have sub-segments of lengths 1 to 10.

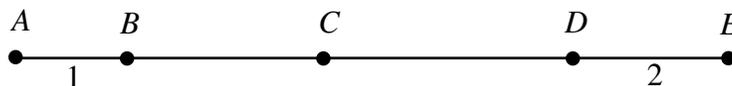
*Solution 2*

Since the longest sub-segment of  $AE$  is  $AE$ , then we must have  $AE = 10$ .

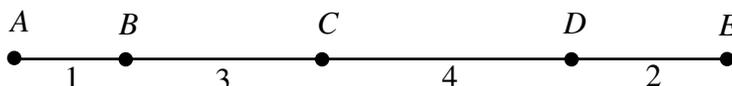
Since the sum of the lengths of the basic sub-segments is the length of  $AE$  and the basic sub-segments are included in the list of sub-segments, then the lengths of the sub-segments must be 1, 2, 3, and 4.

In order to get a sub-segment of length 9, we must have the 2, 3 and 4 adjacent, so the 1 must be on one end.

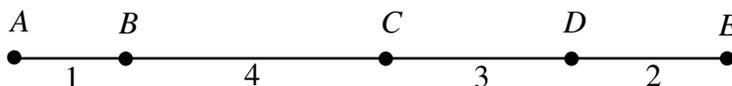
In order to get a sub-segment of length 8, we must have the 1, 3 and 4 adjacent, so the 2 must be on the other end.



This leaves us with two possibilities:



and



In the first possibility, there is no sub-segment of length 5 and two sub-segments of length 4 ( $AC$  and  $CD$ ).

In the second possibility, there is no sub-segment of length 6 and two sub-segments of length 5 ( $AC$  and  $CE$ ).

Having checked all possibilities, we see that it is impossible for the line segment  $AE$  to have sub-segments of lengths 1 to 10.

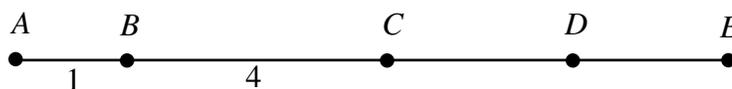
### *Solution 3*

Since the longest sub-segment of  $AE$  is  $AE$ , then we must have  $AE = 10$ .

Since the sum of the lengths of the basic sub-segments is the length of  $AE$  and the basic sub-segments are included in the list of sub-segments, then the lengths of the sub-segments must be 1, 2, 3, and 4.

In order to avoid repeating lengths of sub-segments, we cannot have the 2 or 3 next to the 1.

Therefore, we must have the 1 on one end with the 4 next to it.



But this means that the 2 and the 3 are next to each other, so we have two sub-segments of length 5, which is impossible.

Therefore, it is impossible for the line segment  $AE$  to have sub-segments of lengths 1 to 10.

### (c) *Solution 1*

What happens to the super-sum when we first add a basic sub-segment  $JK$  of length  $\frac{1}{10}$ ? The sub-segments of  $AK$  include all of the sub-segments of  $AJ$  (which have a combined length of 45), and then some additional sub-segments.

The additional sub-segments are ones which are sub-segments of  $AK$  but not of  $AJ$ . In other words, they are sub-segments which include the basic sub-segment  $JK$ . These are the sub-segments  $JK$ ,  $IK$ ,  $HK$ , and so on, all the way to  $BK$  and  $AK$ .

What are the lengths of these additional sub-segments?

$$JK = \frac{1}{10}$$

$$IK = IJ + JK = \frac{1}{9} + \frac{1}{10}$$

$$HK = HI + IJ + JK = \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$$

⋮

$$AK = AB + BC + \cdots + IJ + JK = 1 + \frac{1}{2} + \cdots + \frac{1}{9} + \frac{1}{10}$$

All 10 of these additional sub-segments contain the basic sub-segment  $JK$ , 9 contain  $IJ$ , 8 contain  $HI$ , and so on, with 2 containing  $BC$  and 1 containing  $AB$ .

Therefore, the super-sum of  $AK$  is

$$45 + 10 \left( \frac{1}{10} \right) + 9 \left( \frac{1}{9} \right) + \cdots + 2 \left( \frac{1}{2} \right) + 1(1) = 45 + 10 = 55$$

What happens when we add a basic sub-segment  $KL$  of length  $\frac{1}{11}$ ?

As above, this will add to the sub-segments already contained in  $AK$  an additional 11 sub-segments containing  $KL$ , 10 containing  $JK$ , and so on, with 1 containing  $AB$ .

Thus, the super-sum of  $AL$  will be

$$55 + 11 \left( \frac{1}{11} \right) + 10 \left( \frac{1}{10} \right) + \cdots + 1(1) = 55 + 11 = 66$$

Similarly, as we add the final four basic sub-segments to obtain  $AP$ , we will add 12, then 13, then 14, then 15 to the super-sum.

Therefore, the super-sum of  $AP$  is  $66 + 12 + 13 + 14 + 15 = 120$ .

### *Solution 2*

Each sub-segment of  $AP$  is made up of a number of neighbouring basic sub-segments. We can determine the super-sum of  $AP$  by counting the number of sub-segments in which each basic sub-segment occurs, and so determine the contribution of each basic sub-segment to the super-sum.

The basic sub-segment  $AB$  occurs in sub-segments  $AB, AC, \dots, AP$ , or 15 sub-segments in total.

By symmetry,  $OP$  will also occur in 15 sub-segments.

The basic sub-segment  $BC$  occurs in sub-segments  $BC, BD, \dots, BP$  (14 in total) and  $AC, AD, \dots, AP$  (another 14 in total), for a total of 28 sub-segments.

By symmetry,  $NO$  will also occur in 28 sub-segments.

Is there a better way to count these total number of sub-segments which contain a given basic sub-segment without having to list them all?

Consider  $BC$  again.

A sub-segment containing  $BC$  must have left endpoint  $A$  or  $B$  (2 possibilities) and right endpoint from  $C$  to  $P$  (14 possibilities). Each combination of left endpoint and right endpoint is possible, so there are  $2 \times 14 = 28$  possible sub-segments. (The same argument applies for  $NO$  with 14 possible left endpoints and 2 possible right endpoints.)

We make a table containing each of the remaining basic sub-segments, the number of possible left endpoints for a sub-segment containing it, the number of possible right endpoints, and the total number of sub-segments:

Basic sub-segment	# Possible L endpoints	# Possible R endpoints	Total # of sub-segments
<i>CD</i>	3	13	39
<i>DE</i>	4	12	48
<i>EF</i>	5	11	55
<i>FG</i>	6	10	60
<i>GH</i>	7	9	63
<i>HI</i>	8	8	64
<i>IJ</i>	9	7	63
<i>JK</i>	10	6	60
<i>KL</i>	11	5	55
<i>LM</i>	12	4	48
<i>MN</i>	13	3	39

Therefore, the super-sum is

$$15(1) + 28\left(\frac{1}{2}\right) + 39\left(\frac{1}{3}\right) + 48\left(\frac{1}{4}\right) + 55\left(\frac{1}{5}\right) + 60\left(\frac{1}{6}\right) + 63\left(\frac{1}{7}\right) + 64\left(\frac{1}{8}\right) + 63\left(\frac{1}{9}\right) + 60\left(\frac{1}{10}\right) + 55\left(\frac{1}{11}\right) + 48\left(\frac{1}{12}\right) + 39\left(\frac{1}{13}\right) + 28\left(\frac{1}{14}\right) + 15\left(\frac{1}{15}\right)$$

or

$$15 + 14 + 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

Therefore, the super-sum of  $AP$  is 120.



# Canadian Mathematics Competition

An activity of The Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario

## *2004 Solutions* *Fryer Contest* (Grade 9)

1. (a) Since  $\xrightarrow{R}$  indicates that we take a reciprocal, then  $3 \xrightarrow{R} \frac{1}{3}$ .  
 Since  $\xrightarrow{A}$  indicates to take a reciprocal, then  $\frac{1}{3} \xrightarrow{A} \frac{1}{\frac{1}{3}} + 1 = \frac{4}{3}$ .  
 Continuing,  $\frac{4}{3} \xrightarrow{R} \frac{3}{4}$ ,  $\frac{3}{4} \xrightarrow{A} \frac{3}{4} + 1 = \frac{7}{4}$  and  $\frac{7}{4} \xrightarrow{R} \frac{4}{7}$ .  
 In summary,  $3 \xrightarrow{R} \frac{1}{3} \xrightarrow{A} \frac{4}{3} \xrightarrow{R} \frac{3}{4} \xrightarrow{A} \frac{7}{4} \xrightarrow{R} \frac{4}{7}$ .

(b) As in (a),

$$\begin{aligned} x &\xrightarrow{R} \frac{1}{x} \\ \frac{1}{x} &\xrightarrow{A} \frac{1}{x} + 1 = \frac{1}{x} + \frac{x}{x} = \frac{1+x}{x} \\ \frac{1+x}{x} &\xrightarrow{R} \frac{x}{1+x} \\ \frac{x}{1+x} &\xrightarrow{A} \frac{x}{1+x} + 1 = \frac{x}{1+x} + \frac{1+x}{1+x} = \frac{1+2x}{1+x} \\ \frac{1+2x}{1+x} &\xrightarrow{R} \frac{1+x}{1+2x} \end{aligned}$$

In summary,

$$x \xrightarrow{R} \frac{1}{x} \xrightarrow{A} \frac{1+x}{x} \xrightarrow{R} \frac{x}{1+x} \xrightarrow{A} \frac{1+2x}{1+x} \xrightarrow{R} \frac{1+x}{1+2x}$$

(Since we started by taking the reciprocal of  $x$ , we are assuming that  $x$  is not equal to 0.)

(c) *Solution 1*

By inspection, we observe that if  $\frac{1+x}{1+2x}$  equals  $\frac{14}{27}$ , then  $x + 1 = 14$  or  $x = 13$ . We check to see that the value  $x = 13$  makes the denominator equal to 27, which it does.

(We note that we have to be careful to *check* that our answer works here, because it will not work for instance if  $\frac{1+y}{1+3y} = \frac{2}{5}$ , where the solution is  $y = 3$ .)

*Solution 2*

From (b), if the operations begin with  $x$ , the final result is  $\frac{1+x}{1+2x}$ .

Since we want the final result to be  $\frac{14}{27}$ , then we set  $\frac{1+x}{1+2x} = \frac{14}{27}$  and solve for  $x$  to

determine the original input:

$$\frac{1+x}{1+2x} = \frac{14}{27}$$

$$27(1+x) = 14(1+2x)$$

$$27 + 27x = 14 + 28x$$

$$13 = x$$

Therefore, the input should be 13 to obtain a final result of  $\frac{14}{27}$ . (We can check this by actually doing the 5 operations starting with 13.)

*Solution 3*

We start with  $\frac{14}{27}$  and work backwards through the operations, using the “inverse” of each of the steps.

Since the fifth operation was taking a reciprocal to get  $\frac{14}{27}$ , then the number at the previous step must have been  $\frac{27}{14}$ .

Since the fourth operation was adding 1 to get  $\frac{27}{14}$ , then the number at the previous step must have been  $\frac{27}{14} - 1 = \frac{13}{14}$ .

Since the third operation was taking a reciprocal to get  $\frac{13}{14}$ , then the number at the previous step must have been  $\frac{14}{13}$ .

Since the second operation was adding 1 to get  $\frac{14}{13}$ , then the number at the previous step must have been  $\frac{14}{13} - 1 = \frac{1}{13}$ .

Since the first operation was taking a reciprocal to get  $\frac{1}{13}$ , then the original input must have been 13.

2. (a) Since at least one of each type of prize is given out, then these four prizes account for  $\$5 + \$25 + \$125 + \$625 = \$780$ .

Since there are five prizes given out which total \$905, then the fifth prize must have a value of  $\$905 - \$780 = \$125$ .

Thus, the Fryer Foundation gives out one \$5 prize, one \$25 prize, two \$125 prizes, and one \$625 prize.

- (b) As in (a), giving out one of each type of prize accounts for \$780.

The fifth prize could be a \$5 prize for a total of  $\$780 + \$5 = \$785$ .

The fifth prize could be a \$25 prize for a total of  $\$780 + \$25 = \$805$ .

The fifth prize could be a \$625 prize for a total of  $\$780 + \$625 = \$1405$ .

(We already added an extra \$125 prize in (a).)

- (c) *Solution 1*

Since at least one of each type of prize is given out, this accounts for \$780. So we must figure out how to distribute the remaining  $\$880 - \$780 = \$100$  using at most 5 of each type of prize. We cannot use any \$125 or \$625 prizes, since these are each greater than the remaining amount.

We could use four additional \$25 prizes to make up the \$100.

Could we use fewer than four \$25 prizes? If we use three additional \$25 prizes, this accounts for \$75, which leaves \$25 remaining in \$5 prizes, which can be done by using five additional \$5 prizes.

Could we use fewer than three \$25 prizes? If so, then we would need to make at least \$50 with \$5 prizes, for which we need at least ten such prizes. But we can use at most six \$5 prizes in total, so this is impossible.

Therefore, the two ways of giving out \$880 in prizes under the given conditions are:

- i) one \$625 prize, one \$125 prize, five \$25 prizes, one \$5 prize
- ii) one \$625 prize, one \$125 prize, four \$25 prizes, six \$5 prizes

We can check by addition that each of these totals \$880.

### *Solution 2*

We know that the possible total values using at least one of each type of prize and exactly five prizes are \$785, \$805, \$905 and \$1405.

We try starting with \$785 and \$805 to get to \$880. (Since \$905 and \$1405 are already larger than \$880, we do not need to try these.)

Starting with \$785, we need to give out an additional \$95 to get to \$880. Using three \$25 prizes accounts for \$75, leaving \$20 to be split among four \$5 prizes. (Using fewer than three \$25 prizes will mean we need more than six \$5 prizes in total.) So in this way, we need one \$625 prize, one \$125 prize, four \$25 prizes, and six \$5 prizes (since there were already two included in the \$785).

Starting with \$805, we need to give out an additional \$75 to get to \$880. Using three \$25 prizes will accomplish this, for a total of one \$625 prize, one \$125 prize, five \$25 prizes, and one \$5 prize. We could also use two \$25 prizes and five \$5 prizes to make up the \$75, for a total of one \$625 prize, one \$125 prize, four \$25 prizes, and six \$5 prizes (which is the same as we obtained above starting with \$785). If we use fewer than two additional \$25 prizes, we would need too many \$5 prizes.

Therefore, the two ways of giving out \$880 in prizes under the given conditions are:

- i) one \$625 prize, one \$125 prize, five \$25 prizes, one \$5 prize
- ii) one \$625 prize, one \$125 prize, four \$25 prizes, six \$5 prizes

We can check by addition that each of these totals \$880.

3. (a) If Bob places a 3, then the total of the two numbers so far is 8, so Avril should place a 7 to bring the total up to 15.  
 Since Bob can place a 3 in any the eight empty circles, Avril should place a 7 in the circle directly opposite the one in which Bob places the 3. This allows Avril to win on her next turn.
- (b) As in (a), Bob can place any of the numbers 1, 2, 3, 4, 6, 7, 8, 9 in any of the eight empty circles. On her next turn, Avril should place a disc in the circle directly opposite the one in which Bob put his number. What number should Avril use? Avril should place the number that brings the total up to 15, as shown below:

<u>Bob's First Turn</u>	<u>Total so far</u>	<u>Avril's Second Turn</u>
1	6	9
2	7	8
3	8	7
4	9	6
6	11	4
7	12	3
8	13	2
9	14	1

Since each of these possibilities is available to Avril on her second turn (since 5 is not in the list and none is equal to Bob's number), then she can always win on her second turn.

- (c) Bob can place any of the numbers 1, 2, 4, 5, 7, 8 in any of the six empty circles. We can pair these numbers up so that the sum of the two numbers in the pair plus 6 is equal to 15:  
1 and 8; 2 and 7; 4 and 5

So when Bob uses one of these numbers, Avril can use the other number from the pair, place it directly opposite the one that Bob entered, and the total of the three numbers on this line through the centre will be 15, so Avril will win the game.

4. (a) The 10<sup>th</sup> triangular number is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

The 24<sup>th</sup> triangular number is  $1 + 2 + 3 + \dots + 23 + 24$ . We could add this up by hand or on a calculator to obtain 300. Instead, we could notice that if we pair up the numbers starting with the first and last, then the second and second last, and so on, we obtain:

$$(1 + 24) + (2 + 23) + (3 + 22) + \dots + (11 + 14) + (12 + 13)$$

that is, 12 pairs, each adding to 25, giving a total of  $12 \times 25 = 300$ . (This pairing is the basis for a general method of finding a formula for  $1 + 2 + \dots + n$ .)

- (b) *Solution 1*

Let the three consecutive triangle numbers be

$$1 + 2 + 3 + \dots + (n-1) + n \quad (1),$$

$$1 + 2 + 3 + \dots + (n-1) + n + (n+1) \quad (2), \text{ and}$$

$$1 + 2 + 3 + \dots + (n-1) + n + (n+1) + (n+2) \quad (3).$$

We add these three numbers up, and bring the last term (the  $(n+2)$ ) from (3) to (1):

$$\begin{aligned}
& [1 + 2 + 3 + \cdots + (n-1) + n + (n+2)] \\
& \quad + [1 + 2 + 3 + \cdots + (n-1) + n + (n+1)] \\
& \quad + [1 + 2 + 3 + \cdots + (n-1) + n + (n+1)] \\
& = [1 + 2 + 3 + \cdots + (n-1) + n + (n+1)] + 1 \quad (\text{writing } n+2 \text{ as } (n+1) + 1) \\
& \quad + [1 + 2 + 3 + \cdots + (n-1) + n + (n+1)] \\
& \quad + [1 + 2 + 3 + \cdots + (n-1) + n + (n+1)] \\
& = 3[1 + 2 + 3 + \cdots + (n-1) + n + (n+1)] + 1
\end{aligned}$$

which is 1 more than three times the middle of these three numbers.

*Solution 2*

This solution uses the formula  $1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$  (which we can obtain using a similar pairing argument that we saw in (a)).

So the three consecutive triangle numbers are

$$\begin{aligned}
1 + 2 + 3 + \cdots + (n-1) + n &= \frac{n(n+1)}{2}, \\
1 + 2 + 3 + \cdots + (n-1) + n + (n+1) &= \frac{(n+1)(n+2)}{2}, \text{ and} \\
1 + 2 + 3 + \cdots + (n-1) + n + (n+1) + (n+2) &= \frac{(n+2)(n+3)}{2}
\end{aligned}$$

Adding these three, we obtain

$$\begin{aligned}
& \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} + \frac{(n+2)(n+3)}{2} \\
& = \frac{n^2 + n}{2} + \frac{n^2 + 3n + 2}{2} + \frac{n^2 + 5n + 6}{2} \\
& = \frac{3n^2 + 9n + 8}{2} \\
& = \frac{3n^2 + 9n + 6}{2} + 1 \\
& = 3\left(\frac{n^2 + 3n + 2}{2}\right) + 1
\end{aligned}$$

which is 1 more than three times the middle number of the three consecutive triangular numbers.

(c) *Solution 1*

We are told that the 3<sup>rd</sup>, 6<sup>th</sup> and 8<sup>th</sup> triangular numbers are in arithmetic sequence, and that the 8<sup>th</sup>, 12<sup>th</sup> and 15<sup>th</sup> triangular numbers are in arithmetic sequence. This seems to suggest a pattern, since we add 3 to 3 to get 6 and then 2 to 6 to get 8, followed by 4 to 8 to get 12 and 3 to 12 to get 15.

This suggests that the 15<sup>th</sup>, 20<sup>th</sup> and 24<sup>th</sup> (adding 5 to 15 and then adding 4) triangular numbers are in arithmetic sequence. We know that the 15<sup>th</sup> triangular number is 120 (given) and that the 24<sup>th</sup> triangular number is 300 (from (a)). We can check using a calculator that the 20<sup>th</sup> triangular number is 210. This suggests that the pattern continues. (But this doesn't prove that the pattern works!)

Continuing the pattern we get the following groups of numbers:

24<sup>th</sup>, 30<sup>th</sup> and 35<sup>th</sup>

35<sup>th</sup>, 42<sup>nd</sup> and 48<sup>th</sup>

48<sup>th</sup>, 56<sup>th</sup> and 63<sup>rd</sup>

Let's try the third set:

$$1 + 2 + \cdots + 47 + 48 = (1 + 48) + (2 + 47) + \cdots + (24 + 25) = 24 \times 49 = 1176$$

so this is still too small, since we want all three to be bigger than 2004.

Continuing the pattern:

63<sup>rd</sup>, 72<sup>nd</sup>, 80<sup>th</sup>,

80<sup>th</sup>, 90<sup>th</sup> and 99<sup>th</sup>

Let's try this set. Using our pairing technique from above:

$$1 + 2 + \cdots + 79 + 80 = 40 \times 81 = 3240$$

$$1 + 2 + \cdots + 89 + 90 = 45 \times 91 = 4095$$

$$1 + 2 + \cdots + 98 + 99 = (1 + 2 + \cdots + 99 + 100) - 100 = 50 \times 101 - 100 = 4950$$

Checking these,  $4950 - 4095 = 855 = 4095 - 3240$ , so the 80<sup>th</sup>, 90<sup>th</sup> and 99<sup>th</sup> triangular numbers are in arithmetic sequence.

*Solution 2*

In the two given examples, the difference between the positions of the first two numbers (ie. the 3<sup>rd</sup> and 6<sup>th</sup> in the first example) is one more than the difference between the positions of the second two numbers.

Let's try this pattern and see if we can continue this.

Let's look at the  $n$ th triangular number, the  $(n + 12)$ th triangular number (12 positions further along) and the  $(n + 23)$ th triangular number (11 positions further along). Can we find a value of  $n$  which makes these in arithmetic sequence? (There was no special reason to choose  $(n + 12)$ ; we could try larger or smaller numbers to see if they work.) If these are in arithmetic sequence, then

$$\begin{aligned}
 [1 + 2 + \cdots + (n + 12)] - [1 + 2 + \cdots + n] &= [1 + 2 + \cdots + (n + 23)] - [1 + 2 + \cdots + (n + 12)] \\
 (n + 1) + (n + 2) + \cdots + (n + 12) &= (n + 13) + (n + 14) + \cdots + (n + 23) \\
 12n + (1 + 2 + \cdots + 12) &= 11n + (13 + 14 + \cdots + 23) \\
 n &= (13 + 14 + \cdots + 23) - (1 + 2 + \cdots + 11) - 12 \\
 n &= 11(12) - 12 \\
 n &= 120
 \end{aligned}$$

So the 120<sup>th</sup>, 132<sup>nd</sup> and 143<sup>rd</sup> triangular numbers are in arithmetic progression.

We could calculate these numbers using the pairing idea from (a) to check our answer:

$$1 + 2 + \cdots + 119 + 120 = 60 \times 121 = 7260$$

$$1 + 2 + \cdots + 131 + 132 = 66 \times 133 = 8778$$

$$1 + 2 + \cdots + 142 + 143 = (1 + 2 + \cdots + 143 + 144) - 144 = 72 \times 145 - 144 = 10296$$

Checking these,  $8778 - 7260 = 1518 = 10296 - 8778$ , so the 120<sup>th</sup>, 132<sup>nd</sup> and 143<sup>rd</sup> triangular numbers are in arithmetic sequence.



# Canadian Mathematics Competition

An activity of The Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario

## *2003 Solutions* *Fryer Contest* (Grade 9)

1. (a) *Solution 1*

The average (mean) is equal to the sum of all of the marks, divided by the total number of marks.

Since we know already that there are 32 students (we can check this by looking at the graph), then the average is

$$\frac{1(10) + 2(30) + 2(40) + 1(50) + 4(60) + 6(70) + 9(80) + 4(90) + 3(100)}{32} = \frac{2240}{32} = 70$$

Therefore, the average mark was 70.

*Solution 2*

The average (mean) is equal to the sum of all of the marks, divided by the total number of marks.

Using the bar graph, we can list out all of the marks:

10, 30, 30, 40, 40, 50, 60, 60, 60, 60, 70, 70, 70, 70, 70, 70, 80, 80, 80, 80, 80, 80, 80, 80, 80, 90, 90, 90, 90, 100, 100, 100.

Adding these up using a calculator and dividing by 32, we find that the average mark is 70.

- (b) After his first 6 tests, since Paul's average is 86, then has gotten a total of  $6(86) = 516$  marks.

After getting 100 on his seventh test, Paul has gotten a total of  $516 + 100 = 616$  marks, so his new average is  $\frac{616}{7} = 88$ .

(c) *Solution 1*

If Mary gets 100, her average becomes 90.

If Mary gets 70, her average becomes 87.

So a difference of 30 marks on the test gives a difference of 3 marks in the average.

Since her average is her total number of marks divided by her total number of tests, and a difference of 30 in the total number of marks makes a difference of 3 in her average, then

she will have written  $\frac{30}{3} = 10$  tests.

*Solution 2*

Suppose that after the next test, Mary has written  $n$  tests.

If her average after getting 100 on the next test is 90, then Mary has earned  $90n$  marks in total after the first  $n$  tests, and so  $90n - 100$  before she writes the  $n$ th test.

If her average after getting 70 on the next test is 87, then Mary has gotten  $87n$  marks in total after the  $n$ th test, and so she will have earned  $87n - 70$  marks before the next test.

Therefore, since the number of marks before her next test is the same in either case,

$$87n - 70 = 90n - 100$$

$$30 = 3n$$

$$n = 10$$

So Mary will have written 10 tests.

### *Extension*

We start by using the given information to try to figure out some more things about the marks of the 32 students.

Since the median mark is the “middle mark” in a list of marks which is increasing, then there at least 16 students who have marks that are at least 80.

Since the difference between the highest and lowest marks is 40, and there are students who got at least 80, then the lowest mark in the class cannot be lower than 40.

Since the average mark in the class is 58, then the total number of marks is  $32(58) = 1856$ .

So what does this tell us?

Since at least 16 students got at least 80, then this accounts for at least 1280 marks, leaving  $1856 - 1280 = 576$  marks for the remaining 16 students.

But the lowest possible mark in the class was 40, so these remaining 16 students got at least 40 each, and so got at least  $16(40) = 640$  marks in total!

So we have an inconsistency in the data.

Thus, the teacher made a calculation error.

(There is a variety of different ways of reaching this same conclusion. As before, we know that 16 students will have a mark of at least 80, which accounts for 1280 marks. By the same reasoning, the other 16 students would account for the other 576 marks. The average for these 16 students is thus about 34, which implies that at least one of these lower students must have a mark of 34 or lower. This now contradicts the statement that the range is 40 since  $80 - 34 > 40$ .)

### 2. (a) *Solution 1*

If Xavier goes first and calls 4, then on her turn Yolanda can call any number from 5 to 14, since her number has to be from 1 to 10 greater than Xavier's.

But if Yolanda calls a number from 5 to 14, then Xavier can call 15 on his next turn, since 15 is from 1 to 10 bigger than any of the possible numbers that Yolanda can call.

So Xavier can call 15 on his second turn no matter what Yolanda calls, and is thus always guaranteed to win.

*Solution 2*

If Xavier goes first and calls 4, then Yolanda will call a number of the form  $4 + n$  where  $n$  is a whole number between 1 and 10.

On his second turn, Xavier can call 15 (and thus win) if the difference between 15 and  $4 + n$  is between 1 and 10. But  $15 - (4 + n) = 11 - n$  and since  $n$  is between 1 and 10, then  $11 - n$  is also between 1 and 10, so Xavier can call 15.

Therefore, Xavier's winning strategy is to call 15 on his second turn.

(b) In (a), we saw that if Xavier calls 4, then he can guarantee that he can call 15.

Using the same argument, shifting all of the numbers up, to guarantee that he can call 50, he should call 39 on his previous turn.

(In this case, Yolanda can call any whole number from 40 to 49, and in any of these cases Xavier can call 50, since 50 is no more than 10 greater than any of these numbers.)

In a similar way, to guarantee that he can call 39, he should call 28 on his previous turn, which he can do for the same reasons as above.

To guarantee that he can call 28, he should call 17 on his previous turn.

To guarantee that he can call 17, he should call 6 on his previous turn, which could be his first turn.

Therefore, Xavier's winning strategy is to call 6 on his first turn, 17 on his second turn, 28 on his third turn, 39 on his fourth turn, and 50 on his last turn.

At each step, we are using the fact that Xavier can guarantee that his number on one turn is 11 greater than his number on his previous turn. This is because Yolanda adds 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10 to his previous number, and he can then correspondingly add 10, 9, 8, 7, 6, 5, 4, 3, 2, or 1 to her number, for a total of 11 in each case.

*Extension*

In (b), we discovered that Xavier can always guarantee that the difference between his numbers on two successive turns is 11.

In fact, Yolanda can do the same thing, using exactly the same strategy as Xavier did.

If the target number is between 1 and 9, then Xavier will win on his very first turn by calling that number.

If the target number is then 11 greater than a number between 1 and 9, Xavier will win as in (b). Thus, Xavier wins for 12 through 20.

What about 10 and 11? In each of these cases, Yolanda can win by choosing 10 or 11 on her first turn, which she can do for any initial choice of Xavier's, since he chooses a number between 1 and 9.

Therefore, Yolanda will also win for 21 and 22, and so also for 32 and 33, and so on.

Since either Yolanda or Xavier can repeat their strategy as many times as they want, then Xavier can ensure that he wins if the target number is a multiple of 11 more than one of 1 through 9.

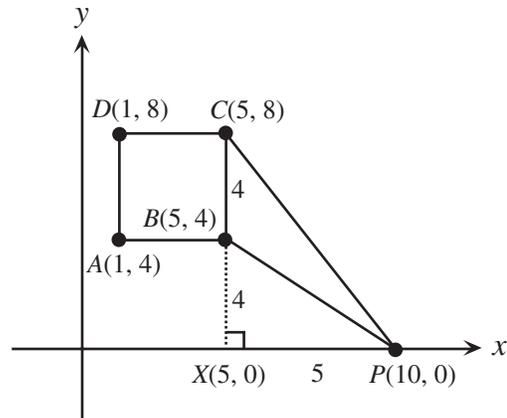
Similarly, Yolanda can ensure that she wins if the target number is a multiple of 11 more than 10 or 11, ie. if the target number is a multiple of 11, or 1 less than a multiple of 11.

3. (a) *Solution 1*

Since  $ABCD$  is a square and  $AD$  has side length 4, then each of the sides of  $ABCD$  has length 4. We can also conclude that  $B$  has coordinates  $(5,4)$  and  $C$  has coordinates  $(5,8)$ . (Since  $AD$  is parallel to the  $y$ -axis, then  $AB$  is parallel to the  $x$ -axis.)

If we turn  $\triangle PBC$  on its side, then we see that its base is  $BC$  which has length 4. Also the height of the triangle is the vertical distance from the line  $BC$  to  $P$ , which is 5. (We can see this by extending the line  $CB$  to a point  $X$  on the  $x$ -axis. Then  $X$  has coordinates  $(5,0)$  and  $PX$ , which has length 5, is perpendicular to  $CB$ .)

Therefore, the area of  $\triangle PBC$  is  $\frac{1}{2}bh = \frac{1}{2}(4)(5) = 10$ .



*Solution 2*

Since  $ABCD$  is a square and  $AD$  has side length 4, then all of the sides of  $ABCD$  have length 4. We can also conclude that  $B$  has coordinates  $(5,4)$  and  $C$  has coordinates  $(5,8)$ . (Since  $AD$  is parallel to the  $y$ -axis, then  $AB$  is parallel to the  $x$ -axis.)

Extend  $CB$  down to a point  $X$  on the  $x$ -axis. Point  $X$  has coordinates  $(5,0)$ .

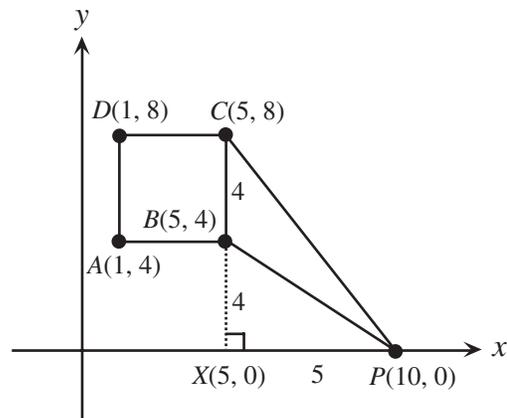
Then the area of  $\triangle PBC$  is the difference between the areas of  $\triangle PCX$  and  $\triangle PBX$ .

$\triangle PCX$  has base  $PX$  of length 5 and height  $CX$  of length 8.

$\triangle PBX$  has base  $PX$  of length 5 and height  $BX$  of length 4.

Therefore, the area of  $\triangle PBC$  is  $\frac{1}{2}(4)(10) - \frac{1}{2}(4)(5) = 10$

as required.



(b) *Solution 1*

Since triangle  $CBE$  lies entirely outside square  $ABCD$ , then the point  $E$  must be “to the right” of the square, ie.  $a$  must be at least 5.

Also, we need to know the area of the square. Since the side length of the square is 4, its area is 16.

Thus if we turn  $\triangle CBE$  on its side, then its we see that its base is  $BC$  which has length 4.

Also the height of the triangle is the vertical distance from the line  $BC$  to  $E$ , which is  $a - 5$  since  $E$  has coordinates  $(a, 0)$  and  $BC$  is part of the line  $x = 5$ .

Therefore, since the area of  $\triangle CBE$  is equal to the area of the square,

$$\frac{1}{2}(4)(a - 5) = 16$$

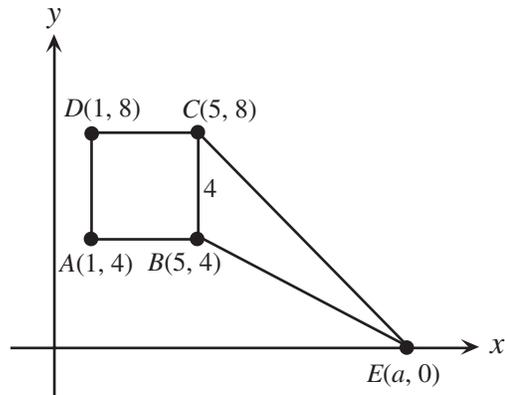
$$2a - 10 = 16$$

$$2a = 26$$

$$a = 13$$

Thus,  $a = 13$ .

(It is easy to verify that if  $a = 13$ , then the height of the triangle is 8 and its base is 4, giving an area of 16.)



### Solution 2

Since triangle  $CBE$  lies entirely outside square  $ABCD$ , then the point  $E$  must be “to the right” of the square, ie.  $a$  must be at least 5.

Also, we need to know the area of the square. Since the side length of the square is 4, its area is 16.

Extend  $CB$  down to a point  $X$  on the  $x$ -axis. Point  $X$  has coordinates  $(5, 0)$ .

Then the area of  $\triangle CBE$  is the difference between the areas of  $\triangle CXE$  and  $\triangle BXE$ .

$\triangle CXE$  has base  $EX$  of length  $a - 5$  and height  $CX$  of length 8.

$\triangle BXE$  has base  $EX$  of length  $a - 5$  and height  $BX$  of length 4.

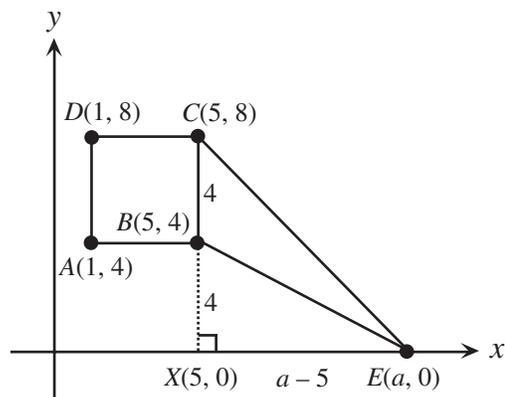
Therefore, since the area of  $\triangle CBE$  is equal to the area of the square,

$$\frac{1}{2}(a - 5)(8) - \frac{1}{2}(a - 5)(4) = 16$$

$$4(a - 5) - 2(a - 5) = 16$$

$$a - 5 = 8$$

$$a = 13$$



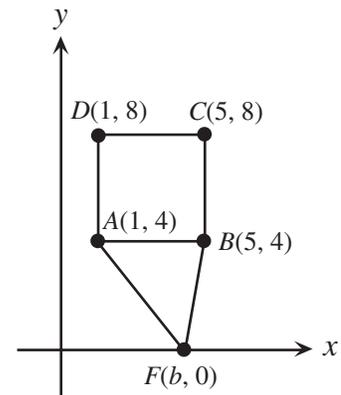
Thus,  $a = 13$ , as required.

(c) Suppose that  $F$  has coordinates  $(b, 0)$ .

Then triangle  $ABF$  has base  $AB$  of length 4.

The height of triangle  $ABF$  is the vertical distance from  $F$  to the line  $AB$ , which is always 4, no matter where  $F$  is.

Thus, the area of triangle  $ABF$  is always  $\frac{1}{2}bh = \frac{1}{2}(4)(4) = 8$ , which is not equal to the area of the square.



### Extension

Since triangle  $DCG$  lies entirely outside the square, then  $G$  is “above” the line through  $D$  and  $C$ , ie. the  $y$ -coordinate of  $G$  is at least 8.

Since the area of triangle  $DCG$  is equal to the area of the square, then the area of triangle  $DCG$  is 16.

Now triangle  $DCB$  has base  $DC$ , which has length 4, so  $\frac{1}{2}bh = \frac{1}{2}(4)h = 16$  or  $h = 8$ .

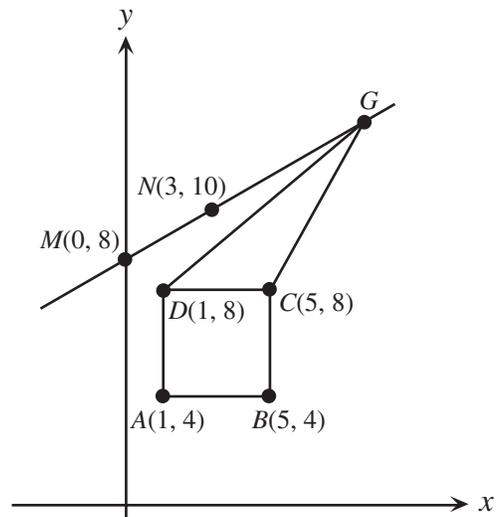
Since the height of triangle  $DCG$  is 8, then  $G$  has  $y$ -coordinate 16, since both  $D$  and  $C$  have  $y$ -coordinate 8.

So we must find the point on the line through  $M(0, 8)$  and  $N(3, 10)$  which has  $y$ -coordinate 16.

To get from  $M$  to  $N$ , we go 3 to the right and up 2.

To get from  $N$  to  $G$ , we go up 6, so we must go 9 to the right.

Therefore,  $G$  has coordinates  $G(12, 16)$ .



4. (a) The best approach here is to list the numbers directly. The possible totals are, from smallest to largest:

1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111

There 15 possible totals, and their sum (that is, the power-sum) is 8888.

(b) *Solution 1*

First, we consider the numbers that are sums of 1 or more of the numbers from  $\{1, 10, 100, 1000\}$ . In (a), we saw that the sum of these numbers is 8888.

What happens when we consider numbers that are sums of 1 or more of the numbers from  $\{1,10,100,1000,10\,000\}$ ? When we do this, we obtain all 15 of the numbers from the previous paragraph, we obtain the 15 numbers obtained by adding 10 000 to all of the numbers from the previous paragraph, and also the number 10 000. (Either our sum does not include 10 000 as a term, or it does; if it doesn't, it must be one of the numbers from (a); if it does, it could be 10 000 on its own, or it could be 10 000 plus one of the numbers from (a).)

Therefore, we have  $15 + 15 + 1 = 31$  numbers in total, whose sum is

$$8888 + [8888 + 15(10\,000)] + 10\,000 = 2(8888) + 160\,000 = 2(88\,888) = 177\,776$$

What happens when we consider the numbers that are sums of 1 or more of the numbers from  $\{1,10,100,1000,10\,000,100\,000\}$ ? When we do this, we obtain all 31 of the numbers from (a), we obtain the 31 numbers obtained by adding 100 000 to all of the numbers from (a), and also the number 100 000.

Therefore, we have  $31 + 31 + 1 = 63$  numbers in total, whose sum is

$$\begin{aligned} 177\,776 + [177\,776 + 31(100\,000)] + 100\,000 &= 2(177\,776) + 3\,200\,000 \\ &= 3\,555\,552 \end{aligned}$$

What happens when we add 1 000 000 to the set? We then obtain, as before,  $63 + 63 + 1 = 127$  numbers in total, whose sum is

$$\begin{aligned} 3\,555\,552 + [3\,555\,552 + 63(1\,000\,000)] + 1\,000\,000 &= 2(3\,555\,552) + 64\,000\,000 \\ &= 71\,111\,104 \end{aligned}$$

Therefore, the power sum is 71 111 104.

### *Solution 2*

There are seven numbers in the given set. When we are considering sums of one or more numbers from the set, each of the seven numbers in the set is either part of the sum, or not part of the sum. So there are two choices (“in” or “out”) for each of the 7 elements.

So we can proceed by first choosing the elements we want to add up, and then adding them up. Since for each of the two possibilities for the “1” (ie. chosen or not chosen), there are two possibilities for the “10”, and there are two possibilities for the “100”, and so on. In total, there will be  $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 = 128$  ways of choosing elements. Notice that this includes the possibility of choosing no elements at all (since we could choose not to select each of the seven elements).

So there are 128 possible sums (including the sum which doesn't add up any numbers at all!).

In how many of these sums is the “1” chosen? If the “1” is chosen, then there are still 2 possibilities for each of the remaining six elements (either chosen or not chosen), so there are  $2^6 = 64$  sums with the 1 included, so the 1 contributes 64 to the power-sum.

In how many of these sums is the “10” chosen? Using exactly the same reasoning, there are 64 sums which include the 10, so the 10 contributes 640 to the power-sum.

Extending this reasoning, each of the 7 elements will contribute to 64 of the sums. (Note that including the “empty” sum doesn’t affect the power-sum.)

Therefore, the power sum is

$$\begin{aligned} &64(1 + 10 + 100 + 1000 + 10000 + 100000 + 1000000) \\ &= 64(1111111) \\ &= 71111104 \end{aligned}$$

### *Extension*

We start looking at small numbers to see if we can see a pattern.

Using the numbers 1, 2, 3, we can form the sums

$$1, 2, 3, 1 + 3 = 4, 1 + 3 = 4, 2 + 3 = 5, 1 + 2 + 3 = 6$$

If we include the 6, we can obtain all of these sums, as well as 6 plus each of these sums.

In other words, we obtain each of the numbers 1 through 12 as totals.

Then including the 12, we can obtain 13 through 24, so we now have each of 1 to 24 as totals.

Including the 24, we obtain all numbers up to 48.

Including the 48, we obtain all numbers up to 96.

Including the 96, we obtain all numbers up to 192.

Therefore, there are 192 different totals possible.

(We can check as well that the sum of the elements in the original set is 192.)