The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

cemc.uwaterloo.ca

Euclid Contest

Tuesday, April 4, 2023
(in North America and South America)

Wednesday, April 5, 2023
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:

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3. For questions marked , place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked , provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions, and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

1. (a) The average of n, $2n$, $3n$, $4n$, $5n$ is 18. What is the value of n?

   (b) Suppose that $2x + y = 5$ and $x + 2y = 7$. What is the average of x and y?

   (c) The average of t^2, $2t$ and 3 is 9. If $t < 0$, determine the value of t.

2. (a) If $Q(5, 3)$ is the midpoint of the line segment with endpoints $P(1, p)$ and $R(r, 5)$, what are the values of p and r?

   (b) A line with slope 3 and another line with slope -1 intersect at $P(3, 6)$. What is the distance between the x-intercepts of the two lines?

   (c) For some value of t, the line with equation $y = tx + t$ is perpendicular to the line with equation $y = 2x + 7$. Determine the point of intersection of these two lines.

3. (a) The positive divisors of 6 are 1, 2, 3, and 6. What is the sum of the positive divisors of 64?

   (b) Fionn wrote 4 consecutive integers on a whiteboard. Lexi came along and erased one of the integers. Fionn noticed that the sum of the remaining integers was 847. What integer did Lexi erase?

   (c) An arithmetic sequence with 7 terms has first term d^2 and common difference d. The sum of the 7 terms in the sequence is 756. Determine all possible values of d.

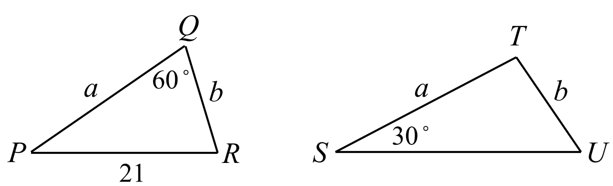
   (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.)

4. (a) Liang and Edmundo paint at different but constant rates. Liang can paint a room in 3 hours if she works alone. Edmundo can paint the same room in 4 hours if he works alone. Liang works alone for 2 hours and then stops. Edmundo finishes painting the room. How many minutes will Edmundo need to finish painting the room?

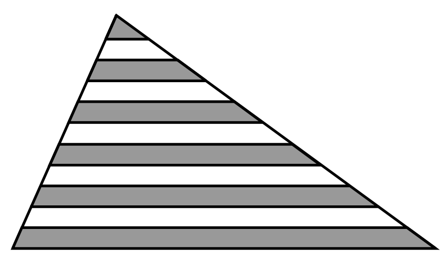
(b) On January 1, 2021, an investment had a value of \$400.
From January 1, 2021 to January 1, 2022, the value of the investment increased by $A\%$ from its value on January 1, 2021 for some $A > 0$.
From January 1, 2022 to January 1, 2023, the value of the investment decreased by $A\%$ from its value on January 1, 2022.
On January 1, 2023, the value of the investment was \$391.
Determine all possible values of A.

5. (a) Suppose that $f(x) = x^2 + (2n-1)x + (n^2 - 22)$ for some integer n. What is the smallest positive integer n for which $f(x)$ has no real roots?

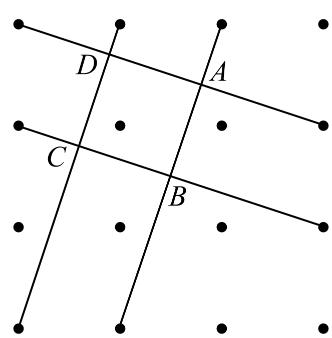
(b) In the diagram, $\triangle PQR$ has $PQ = a$, $QR = b$, $PR = 21$, and $\angle PQR = 60°$. Also, $\triangle STU$ has $ST = a$, $TU = b$, $\angle TSU = 30°$, and $\sin(\angle TUS) = \frac{4}{5}$. Determine the values of a and b.



6. (a) A triangle of area $770\ \mathrm{cm}^2$ is divided into 11 regions of equal height by 10 lines that are all parallel to the base of the triangle. Starting from the top of the triangle, every other region is shaded, as shown. What is the total area of the shaded regions?



(b) A square lattice of 16 points is constructed such that the horizontal and vertical distances between adjacent points are all exactly 1 unit. Each of four pairs of points are connected by a line segment, as shown. The intersections of these line segments are the vertices of square $ABCD$. Determine the area of square $ABCD$.



7. (a) A bag contains 3 red marbles and 6 blue marbles. Akshan removes one marble at a time until the bag is empty. Each marble that they remove is chosen randomly from the remaining marbles. Given that the first marble that Akshan removes is red and the third marble that they remove is blue, what is the probability that the last two marbles that Akshan removes are both blue?

(b) Determine the number of quadruples of positive integers (a, b, c, d) with $a < b < c < d$ that satisfy both of the following system of equations:

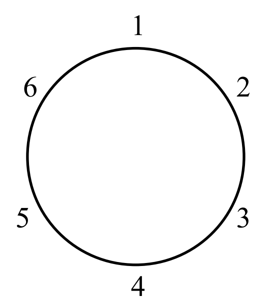
$$\begin{aligned} ac + ad + bc + bd &= 2023 \\ a + b + c + d &= 296 \end{aligned}$$

8. (a) Suppose that $\triangle ABC$ is right-angled at B and has $AB = n(n + 1)$ and $AC = (n + 1)(n + 4)$, where n is a positive integer. Determine the number of positive integers $n < 100\,000$ for which the length of side BC is also an integer.

(b) Determine all real values of x for which

$$\sqrt{\log_2 x \cdot \log_2(4x) + 1} + \sqrt{\log_2 x \cdot \log_2\left(\frac{x}{64}\right) + 9} = 4$$

9. At the Canadian Eatery with Multiple Configurations, there are round tables, around which chairs are placed. When a table has n chairs around it for some integer $n \geq 3$, the chairs are labelled 1, 2, 3, $\ldots$, $n - 1$, n in order around the table. A table is considered full if no more people can be seated without having two people sit in neighbouring chairs. For example, when $n = 6$, full tables occur when people are seated in chairs labelled $\{1, 4\}$ or $\{2, 5\}$ or $\{3, 6\}$ or $\{1, 3, 5\}$ or $\{2, 4, 6\}$. Thus, there are 5 different full tables when $n = 6$.



(a) Determine all ways in which people can be seated around a table with 8 chairs so that the table is full, in each case giving the labels on the chairs in which people are sitting.

(b) A full table with $6k + 5$ chairs, for some positive integer k, has t people seated in its chairs. Determine, in terms of k, the number of possible values of t.

(c) Determine the number of different full tables when $n = 19$.

10. For every real number x, define $\lfloor x \rfloor$ to be equal to the greatest integer less than or equal to x. (We call this the "floor" of x.) For example, $\lfloor 4.2 \rfloor = 4$, $\lfloor 5.7 \rfloor = 5$, $\lfloor -3.4 \rfloor = -4$, $\lfloor 0.4 \rfloor = 0$, and $\lfloor 2 \rfloor = 2$.

(a) Determine the integer equal to $\left\lfloor \frac{1}{3} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor + \left\lfloor \frac{3}{3} \right\rfloor + \ldots + \left\lfloor \frac{59}{3} \right\rfloor + \left\lfloor \frac{60}{3} \right\rfloor$.

(The sum has 60 terms.)

(b) Determine a polynomial $p(x)$ so that for every positive integer $m > 4$,

$$\lfloor p(m) \rfloor = \left\lfloor \frac{1}{3} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor + \left\lfloor \frac{3}{3} \right\rfloor + \ldots + \left\lfloor \frac{m-2}{3} \right\rfloor + \left\lfloor \frac{m-1}{3} \right\rfloor$$

(The sum has $m - 1$ terms.)

A *polynomial* $f(x)$ is an algebraic expression of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ for some integer $n \geq 0$ and for some real numbers $a_n, a_{n-1}, \ldots, a_1, a_0$.

(c) For each integer $n \geq 1$, define $f(n)$ to be equal to an infinite sum:

$$f(n) = \left\lfloor \frac{n}{1^2+1} \right\rfloor + \left\lfloor \frac{2n}{2^2+1} \right\rfloor + \left\lfloor \frac{3n}{3^2+1} \right\rfloor + \left\lfloor \frac{4n}{4^2+1} \right\rfloor + \left\lfloor \frac{5n}{5^2+1} \right\rfloor + \cdots$$

(The sum contains the terms $\left\lfloor \frac{kn}{k^2+1} \right\rfloor$ for all positive integers k, and no other terms.)

Suppose $f(t+1) - f(t) = 2$ for some odd positive integer t. Prove that t is a prime number.

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- Information about careers in and applications of mathematics and computer science

For teachers...

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- Obtain information about our 2023/2024 contests
- Look at our free online courseware for high school students
- Learn about our face-to-face workshops and our web resources
- Subscribe to our free Problem of the Week
- Investigate our online Master of Mathematics for Teachers
- Find your school's contest results

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1. (a) What is the value of $\dfrac{3^2 - 2^3}{2^3 - 3^2}$?

   (b) What is the value of $\sqrt{\sqrt{81} + \sqrt{9} - \sqrt{64}}$?

   (c) Determine all real numbers x for which $\dfrac{1}{\sqrt{x^2 + 7}} = \dfrac{1}{4}$.

2. (a) Find the three ordered pairs of integers (a, b) with $1 < a < b$ and $ab = 2022$.

   (b) Suppose that c and d are integers with $c > 0$ and $d > 0$ and $\dfrac{2c + 1}{2d + 1} = \dfrac{1}{17}$. What is the smallest possible value of d?

   (c) Suppose that p, r and t are real numbers for which $(px + r)(x + 5) = x^2 + 3x + t$ is true for all real numbers x. Determine the value of t.

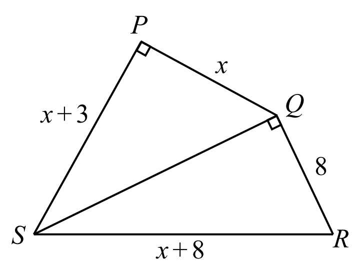
3. (a) A large water jug is $\frac{1}{4}$ full of water. After 24 litres of water are added, the jug is $\frac{5}{8}$ full. What is the volume of the jug, in litres?

(b) Stephanie starts with a large number of soccer balls. She gives $\frac{2}{5}$ of them to Alphonso and $\frac{6}{11}$ of them to Christine. The number of balls that she is left with is a multiple of 9. What is the smallest number of soccer balls with which Stephanie could have started?

(c) Each student in a math club is in either the Junior section or the Senior section. No student is in both sections.
Of the Junior students, 60% are left-handed and 40% are right-handed.
Of the Senior students, 10% are left-handed and 90% are right-handed.
No student in the math club is both left-handed and right-handed.
The total number of left-handed students is equal to the total number of right-handed students in the math club.
Determine the percentage of math club members that are in the Junior section.

4. (a) Hexagon $ABCDEF$ has vertices $A(0,0)$, $B(4,0)$, $C(7,2)$, $D(7,5)$, $E(3,5)$, $F(0,3)$. What is the area of hexagon $ABCDEF$?

(b) In the diagram, $\triangle PQS$ is right-angled at P and $\triangle QRS$ is right-angled at Q. Also, $PQ = x$, $QR = 8$, $RS = x + 8$, and $SP = x + 3$ for some real number x. Determine all possible values of the perimeter of quadrilateral $PQRS$.



5. (a) A list a_1, a_2, a_3, a_4 of rational numbers is defined so that if one term is equal to r, then the next term is equal to $1 + \dfrac{1}{1+r}$. For example, if $a_3 = \dfrac{41}{29}$, then $a_4 = 1 + \dfrac{1}{1 + (41/29)} = \dfrac{99}{70}$. If $a_3 = \dfrac{41}{29}$, what is the value of a_1?

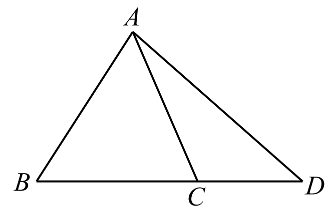
(b) A hollow cylindrical tube has a radius of 10 mm and a height of 100 mm. The tube sits flat on one of its circular faces on a horizontal table. The tube is filled with water to a depth of h mm. A solid cylindrical rod has a radius of 2.5 mm and a height of 150 mm. The rod is inserted into the tube so that one of its circular faces sits flat on the bottom of the tube. The height of the water in the tube is now 64 mm. Determine the value of h.

6\. (a) A function f has the property that $f\left(\dfrac{2x+1}{x}\right) = x + 6$ for all real values of $x \neq 0$. What is the value of $f(4)$?

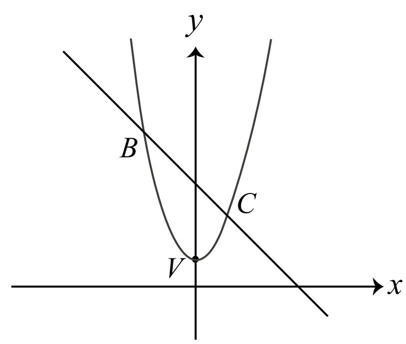
(b) Determine all real numbers a, b and c for which the graph of the function $y = \log_a(x+b) + c$ passes through the points $P(3,5)$, $Q(5,4)$ and $R(11,3)$.

7\. (a) A computer is programmed to choose an integer between 1 and 99, inclusive, so that the probability that it selects the integer x is equal to $\log_{100}\left(1 + \dfrac{1}{x}\right)$. Suppose that the probability that $81 \leq x \leq 99$ is equal to 2 times the probability that $x = n$ for some integer n. What is the value of n?

(b) In the diagram, $\triangle ABD$ has C on BD. Also, $BC = 2$, $CD = 1$, $\dfrac{AC}{AD} = \dfrac{3}{4}$, and $\cos(\angle ACD) = -\dfrac{3}{5}$. Determine the length of AB.



8\. (a) Suppose that $a > \frac{1}{2}$ and that the parabola with equation $y = ax^2 + 2$ has vertex V. The parabola intersects the line with equation $y = -x + 4a$ at points B and C, as shown. If the area of $\triangle VBC$ is $\frac{72}{5}$, determine the value of a.



(b) Consider the following statement:

> There is a triangle that is not equilateral whose side lengths form a geometric sequence, and the measures of whose angles form an arithmetic sequence.

Show that this statement is true by finding such a triangle or prove that it is false by demonstrating that there cannot be such a triangle.

9. Suppose that m and n are positive integers with $m \geq 2$. The (m,n)-*sawtooth sequence* is a sequence of consecutive integers that starts with 1 and has n teeth, where each tooth starts with 2, goes up to m and back down to 1. For example, the $(3,4)$-sawtooth sequence is

$$\begin{array}{ccccccccccccccccc}
 & & 3 & & & & 3 & & & & 3 & & & & 3 & & \\
 & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & \\
1 & & & & 1 & & & & 1 & & & & 1 & & & & 1
\end{array}$$

The $(3,4)$-sawtooth sequence includes 17 terms and the average of these terms is $\frac{33}{17}$.

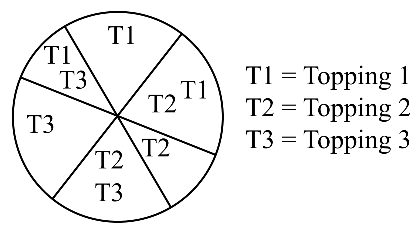
(a) Determine the sum of the terms in the $(4,2)$-sawtooth sequence.

(b) For each positive integer $m \geq 2$, determine a simplified expression for the sum of the terms in the $(m,3)$-sawtooth sequence.

(c) Determine all pairs (m,n) for which the sum of the terms in the (m,n)-sawtooth sequence is 145.

(d) Prove that, for all pairs of positive integers (m,n) with $m \geq 2$, the average of the terms in the (m,n)-sawtooth sequence is not an integer.

10. At Pizza by Alex, toppings are put on circular pizzas in a random way. Every topping is placed on a randomly chosen semi-circular half of the pizza and each topping's semi-circle is chosen independently. For each topping, Alex starts by drawing a diameter whose angle with the horizonal is selected uniformly at random. This divides the pizza into two semi-circles. One of the two halves is then chosen at random to be covered by the topping.



(a) For a 2-topping pizza, determine the probability that at least $\frac{1}{4}$ of the pizza is covered by both toppings.

(b) For a 3-topping pizza, determine the probability that some region of the pizza with non-zero area is covered by all 3 toppings. (The diagram above shows an example where no region is covered by all 3 toppings.)

(c) Suppose that N is a positive integer. For an N-topping pizza, determine the probability, in terms of N, that some region of the pizza with non-zero area is covered by all N toppings.

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1. (a) What is the value of a for which $(a-1)+(2a-3)=14$?

   (b) What are the two values of c for which $(c^2-c)+(2c-3)=9$?

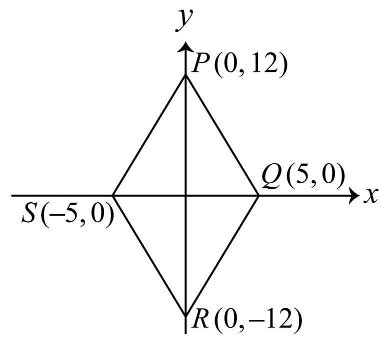
   (c) Determine all values of x for which $\dfrac{1}{x^2}+\dfrac{3}{2x^2}=10$.

2. (a) What is the sum of the digits of the integer equal to $(10^3+1)^2$?

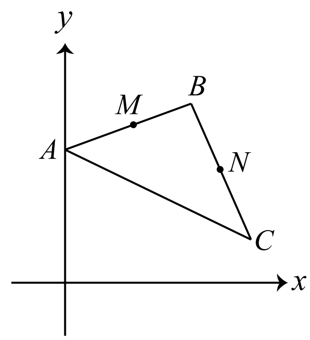
   (b) A bakery sells small and large cookies. Before a price increase, the price of each small cookie is \$1.50 and the price of each large cookie is \$2.00. The price of each small cookie is increased by 10% and the price of each large cookie is increased by 5%. What is the percentage increase in the total cost of a purchase of 2 small cookies and 1 large cookie?

   (c) Qing is twice as old as Rayna. Qing is 4 years younger than Paolo. The average age of Paolo, Qing and Rayna is 13. Determine their ages.

3. (a) In the diagram, $PQRS$ is a quadrilateral. What is its perimeter?



(b) In the diagram, A has coordinates $(0, 8)$. Also, the midpoint of AB is $M(3, 9)$ and the midpoint of BC is $N(7, 6)$. What is the slope of AC?



(c) The parabola with equation $y = -2x^2 + 4x + c$ has vertex $V(1, 18)$. The parabola intersects the y-axis at D and the x-axis at E and F. Determine the area of $\triangle DEF$.

4. (a) If $3(8^x) + 5(8^x) = 2^{61}$, what is the value of the real number x?

(b) For some real numbers m and n, the list $3n^2$, m^2, $2(n + 1)^2$ consists of three consecutive integers written in increasing order. Determine all possible values of m.

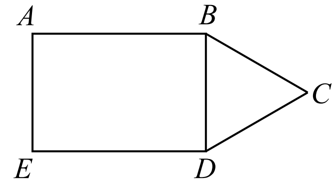
5. (a) Chinara starts with the point $(3, 5)$, and applies the following three-step process, which we call $\mathcal{P}$:

Step 1: Reflect the point in the x-axis.
Step 2: Translate the resulting point 2 units upwards.
Step 3: Reflect the resulting point in the y-axis.

As she does this, the point $(3, 5)$ moves to $(3, -5)$, then to $(3, -3)$, and then to $(-3, -3)$.

Chinara then starts with a different point S_0. She applies the three-step process $\mathcal{P}$ to the point S_0 and obtains the point S_1. She then applies $\mathcal{P}$ to S_1 to obtain the point S_2. She applies $\mathcal{P}$ four more times, each time using the previous output of $\mathcal{P}$ to be the new input, and eventually obtains the point $S_6(-7, -1)$. What are the coordinates of the point S_0?

(b) In the diagram, $ABDE$ is a rectangle, $\triangle BCD$ is equilateral, and AD is parallel to BC. Also, $AE = 2x$ for some real number x.



(i) Determine the length of AB in terms of x.

(ii) Determine positive integers r and s for which $\dfrac{AC}{AD} = \sqrt{\dfrac{r}{s}}$.

6. (a) Suppose that $n > 5$ and that the numbers $t_1, t_2, t_3, \ldots, t_{n-2}, t_{n-1}, t_n$ form an arithmetic sequence with n terms. If $t_3 = 5$, $t_{n-2} = 95$, and the sum of all n terms is 1000, what is the value of n?

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.)

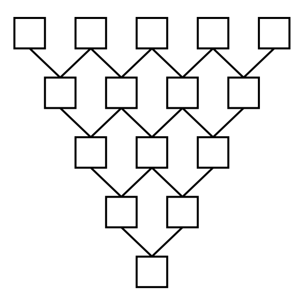
(b) Suppose that a and r are real numbers. A geometric sequence with first term a and common ratio r has 4 terms. The sum of this geometric sequence is $6 + 6\sqrt{2}$. A second geometric sequence has the same first term a and the same common ratio r, but has 8 terms. The sum of this second geometric sequence is $30 + 30\sqrt{2}$. Determine all possible values for a.

(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the common ratio. For example, $3, -6, 12, -24$ are the first four terms of a geometric sequence.)

7. (a) A bag contains 3 green balls, 4 red balls, and no other balls. Victor removes balls randomly from the bag, one at a time, and places them on a table. Each ball in the bag is equally likely to be chosen each time that he removes a ball. He stops removing balls when there are two balls of the same colour on the table. What is the probability that, when he stops, there is at least 1 red ball and at least 1 green ball on the table?

(b) Suppose that $f(a) = 2a^2 - 3a + 1$ for all real numbers a and $g(b) = \log_{\frac{1}{2}} b$ for all $b > 0$. Determine all θ with $0 \leq \theta \leq 2\pi$ for which $f(g(\sin\theta)) = 0$.

8. (a) Five distinct integers are to be chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and placed in some order in the top row of boxes in the diagram. Each box that is not in the top row then contains the product of the integers in the two boxes connected to it in the row directly above. Determine the number of ways in which the integers can be chosen and placed in the top row so that the integer in the bottom box is 9 953 280 000.



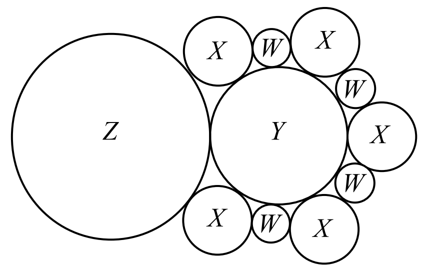
(b) Prove that the integer $\dfrac{(1!)(2!)(3!)\cdots(398!)(399!)(400!)}{200!}$ is a perfect square. (In this fraction, the numerator is the product of the factorials of the integers from 1 to 400, inclusive.)

9. (a) Suppose that $a = 5$ and $b = 4$. Determine all pairs of integers (K, L) for which $K^2 + 3L^2 = a^2 + b^2 - ab$.

   (b) Prove that, for all integers K and L, there is at least one pair of integers (a, b) for which $K^2 + 3L^2 = a^2 + b^2 - ab$.

   (c) Prove that, for all integers a and b, there is at least one pair of integers (K, L) for which $K^2 + 3L^2 = a^2 + b^2 - ab$.

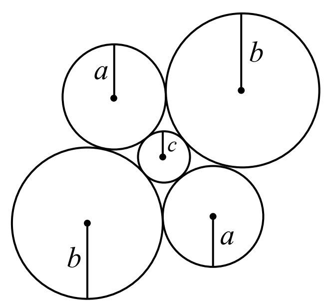
10. (a) In the diagram, eleven circles of four different sizes are drawn. Each circle labelled W has radius 1, each circle labelled X has radius 2, the circle labelled Y has radius 4, and the circle labelled Z has radius r. Each of the circles labelled W or X is tangent to three other circles. The circle labelled Y is tangent to all ten of the other circles. The circle labelled Z is tangent to three other circles. Determine positive integers s and t for which $r = \dfrac{s}{t}$.



   (b) Suppose that c is a positive integer. Define $f(c)$ to be the number of pairs (a, b) of positive integers with $c < a < b$ for which two circles of radius a, two circles of radius b, and one circle of radius c can be drawn so that

   - each circle of radius a is tangent to both circles of radius b and to the circle of radius c, and
   - each circle of radius b is tangent to both circles of radius a and to the circle of radius c,

   as shown. Determine all positive integers c for which $f(c)$ is even.



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For students...

Thank you for writing the 2021 Euclid Contest! Each year, more than 260 000 students from more than 80 countries register to write the CEMC's Contests.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2021 Canadian Senior Mathematics Contest, which will be written in November 2021.

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Visit our website `cemc.uwaterloo.ca` to

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- Look at our free online courseware for high school students
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The CENTRE for EDUCATION
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Euclid Contest

Tuesday, April 7, 2020
(in North America and South America)

Wednesday, April 8, 2020
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked, place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked, provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions, and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

1. (a) If $x = 11$, what is the value of $\dfrac{3x+6}{x+2}$?

   (b) What is the y-intercept of the line that passes through $A(-1, 5)$ and $B(1, 7)$?

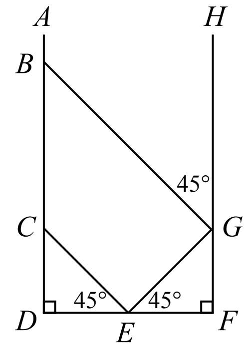
   (c) The lines with equations $y = 3x + 7$, $y = x + 9$, and $y = mx + 17$ intersect at a single point. Determine the value of m.

2. (a) The three-digit positive integer m is odd and has three distinct digits. If the hundreds digit of m equals the product of the tens digit and ones (units) digit of m, what is m?

   (b) Eleanor has 100 marbles, each of which is black or gold. The ratio of the number of black marbles to the number of gold marbles is $1 : 4$. How many gold marbles should she add to change this ratio to $1 : 6$?

   (c) Suppose that n is a positive integer and that the value of $\dfrac{n^2+n+15}{n}$ is an integer. Determine all possible values of n.

3. (a) Donna has a laser at C. She points the laser beam at the point E. The beam reflects off of DF at E and then off of FH at G, as shown, arriving at point B on AD. If $DE = EF = 1$ m, what is the length of BD, in metres?



(b) Ada starts with $x = 10$ and $y = 2$, and applies the following process:

Step 1: Add x and y. Let x equal the result. The value of y does not change.

Step 2: Multiply x and y. Let x equal the result. The value of y does not change.

Step 3: Add y and 1. Let y equal the result. The value of x does not change.

Ada keeps track of the values of x and y:

	x	y
Before Step 1	10	2
After Step 1	12	2
After Step 2	24	2
After Step 3	24	3

Continuing now with $x = 24$ and $y = 3$, Ada applies the process two more times. What is the final value of x?

(c) Determine all integers k, with $k \neq 0$, for which the parabola with equation $y = kx^2 + 6x + k$ has two distinct x-intercepts.

4. (a) The positive integers a and b have no common divisor larger than 1. If the difference between b and a is 15 and $\frac{5}{9} < \frac{a}{b} < \frac{4}{7}$, what is the value of $\frac{a}{b}$?

(b) A geometric sequence has first term 10 and common ratio $\frac{1}{2}$.
An arithmetic sequence has first term 10 and common difference d.
The ratio of the 6th term in the geometric sequence to the 4th term in the geometric sequence equals the ratio of the 6th term in the arithmetic sequence to the 4th term in the arithmetic sequence.
Determine all possible values of d.

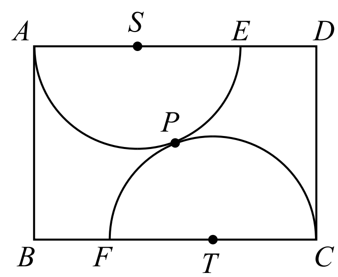
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.
A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the common ratio. For example, 3, 6, 12 is a geometric sequence with three terms.)

5. (a) For each positive real number x, define $f(x)$ to be the number of prime numbers p that satisfy $x \leq p \leq x + 10$. What is the value of $f(f(20))$?

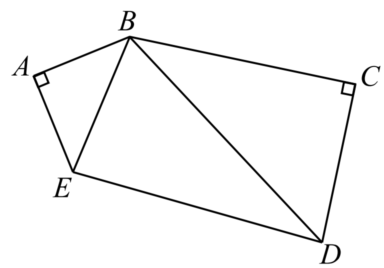
(b) Determine all triples (x, y, z) of real numbers that satisfy the following system of equations:

$$\begin{aligned}(x-1)(y-2) &= 0\\(x-3)(z+2) &= 0\\x + yz &= 9\end{aligned}$$

6. (a) Rectangle $ABCD$ has $AB = 4$ and $BC = 6$. The semi-circles with diameters AE and FC each have radius r, have centres S and T, and touch at a single point P, as shown. What is the value of r?



(b) In the diagram, $\triangle ABE$ is right-angled at A, $\triangle BCD$ is right-angled at C, $\angle ABC = 135^\circ$, and $AB = AE = 7\sqrt{2}$. If $DC = 4x$, $DB = 8x$ and $DE = 8x - 6$ for some real number x, determine all possible values of x.



7. (a) Suppose that the function g satisfies $g(x) = 2x - 4$ for all real numbers x and that g^{-1} is the inverse function of g. Suppose that the function f satisfies $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$ for all real numbers x. What is the value of $f(\pi)$?

(b) Determine all pairs of angles (x, y) with $0^\circ \leq x < 180^\circ$ and $0^\circ \leq y < 180^\circ$ that satisfy the following system of equations:

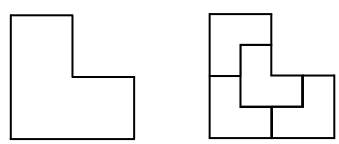
$$\log_2(\sin x \cos y) = -\frac{3}{2}$$

$$\log_2\left(\frac{\sin x}{\cos y}\right) = \frac{1}{2}$$

8. (a) Four tennis players Alain, Bianca, Chen, and Dave take part in a tournament in which a total of three matches are played. First, two players are chosen randomly to play each other. The other two players also play each other. The winners of the two matches then play to decide the tournament champion. Alain, Bianca and Chen are equally matched (that is, when a match is played between any two of them, the probability that each player wins is $\frac{1}{2}$). When Dave plays each of Alain, Bianca and Chen, the probability that Dave wins is p, for some real number p. Determine the probability that Bianca wins the tournament, expressing your answer in the form $\dfrac{ap^2 + bp + c}{d}$ where a, b, c, and d are integers.

(b) Three microphones A, B and C are placed on a line such that A is 1 km west of B and C is 2 km east of B. A large explosion occurs at a point P not on this line. Each of the three microphones receives the sound. The sound travels at $\frac{1}{3}$ km/s. Microphone B receives the sound first, microphone A receives the sound $\frac{1}{2}$ s later, and microphone C receives it 1 s after microphone A. Determine the distance from microphone B to the explosion at P.

9. (a) An L shape is made by adjoining three congruent squares. The L is subdivided into four smaller L shapes, as shown. Each of the resulting L's is subdivided in this same way. After the third round of subdivisions, how many L's of the smallest size are there?



Original L   After 1 round

(b) After the third round of subdivisions, how many L's of the smallest size are in the same orientation as the original L?

(c) Starting with the original L shape, 2020 rounds of subdivisions are made. Determine the number of L's of the smallest size that are in the same orientation as the original L.

10. Kerry has a list of n integers $a_1, a_2, \ldots, a_n$ satisfying $a_1 \leq a_2 \leq \ldots \leq a_n$. Kerry calculates the pairwise sums of all $m = \frac{1}{2}n(n-1)$ possible pairs of integers in her list and orders these pairwise sums as $s_1 \leq s_2 \leq \ldots \leq s_m$. For example, if Kerry's list consists of the three integers $1, 2, 4$, the three pairwise sums are $3, 5, 6$.

(a) Suppose that $n = 4$ and that the 6 pairwise sums are $s_1 = 8$, $s_2 = 104$, $s_3 = 106$, $s_4 = 110$, $s_5 = 112$, and $s_6 = 208$. Determine two possible lists a_1, a_2, a_3, a_4 that Kerry could have.

(b) Suppose that $n = 5$ and that the 10 pairwise sums are $s_1, s_2, \ldots, s_{10}$. Prove that there is only one possibility for Kerry's list a_1, a_2, a_3, a_4, a_5.

(c) Suppose that $n = 16$. Prove that there are two different lists $a_1, a_2, \ldots, a_{16}$ and $b_1, b_2, \ldots, b_{16}$ that produce the same list of sums $s_1, s_2, \ldots, s_{120}$.

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The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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Euclid Contest

Wednesday, April 3, 2019
(in North America and South America)

Thursday, April 4, 2019
(outside of North America and South America)

UNIVERSITY OF
WATERLOO

Time: $2\frac{1}{2}$ hours

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Number of questions: 10 **Each question is worth 10 marks**

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   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

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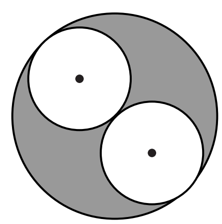
A Note about Bubbling
Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

1. (a) Joyce has two identical jars. The first jar is $\frac{3}{4}$ full of water and contains 300 mL of water. The second jar is $\frac{1}{4}$ full of water. How much water, in mL, does the second jar contain?

   (b) What integer a satisfies $3 < \dfrac{24}{a} < 4$?

   (c) If $\dfrac{1}{x^2} - \dfrac{1}{x} = 2$, determine all possible values of x.

2. (a) In the diagram, two small circles of radius 1 are tangent to each other and to a larger circle of radius 2. What is the area of the shaded region?



   (b) Kari jogs at a constant speed of 8 km/h. Mo jogs at a constant speed of 6 km/h. Kari and Mo jog from the same starting point to the same finishing point along a straight road. Mo starts at 10:00 a.m. Kari and Mo both finish at 11:00 a.m. At what time did Kari start to jog?

   (c) The line with equation $x+3y = 7$ is parallel to the line with equation $y = mx+b$. The line with equation $y = mx + b$ passes through the point $(9, 2)$. Determine the value of b.

3. (a) Michelle calculates the average of the following numbers:

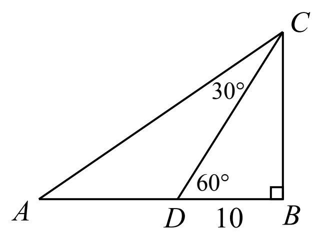
$$5, 10, 15, 16, 24, 28, 33, 37$$

Daphne removes one number and calculates the average of the remaining numbers. The average that Daphne calculates is one less than the average that Michelle calculates. Which number does Daphne remove?

(b) If $16^{\frac{15}{x}} = 32^{\frac{4}{3}}$, what is the value of x?

(c) Suppose that $\dfrac{2^{2022} + 2^a}{2^{2019}} = 72$. Determine the value of a.

4. (a) In the diagram, $\triangle ABC$ has a right angle at B and point D lies on AB. If $DB = 10$, $\angle ACD = 30^\circ$ and $\angle CDB = 60^\circ$, what is the length of AD?



(b) The points $A(d, -d)$ and $B(-d + 12, 2d - 6)$ both lie on a circle centered at the origin. Determine the possible values of d.

5. (a) Determine the two pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a} + \sqrt{b} = \sqrt{50}$.

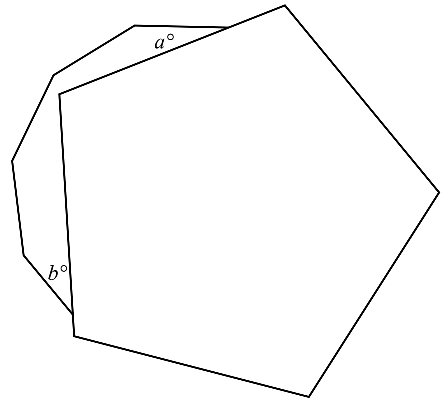
(b) Consider the system of equations:

$$c + d = 2000$$
$$\frac{c}{d} = k$$

Determine the number of integers k with $k \geq 0$ for which there is at least one pair of integers (c, d) that is a solution to the system.

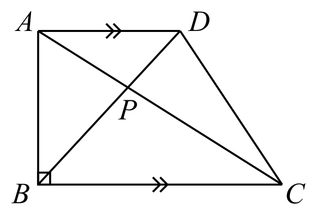
6. (a) A regular pentagon covers part of another regular polygon, as shown. This regular polygon has n sides, five of which are completely or partially visible. In the diagram, the sum of the measures of the angles marked a° and b° is 88°. Determine the value of n.

(The side lengths of a *regular polygon* are all equal, as are the measures of its interior angles.)



(b) In trapezoid $ABCD$, BC is parallel to AD and BC is perpendicular to AB. Also, the lengths of AD, AB and BC, in that order, form a geometric sequence. Prove that AC is perpendicular to BD.

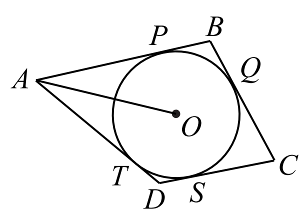
(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant.)



7. (a) Determine all real numbers x for which $2\log_2(x-1) = 1 - \log_2(x+2)$.

(b) Consider the function $f(x) = x^2 - 2x$. Determine all real numbers x that satisfy the equation $f(f(f(x))) = 3$.

8. (a) A circle has centre O and radius 1. Quadrilateral $ABCD$ has all 4 sides tangent to the circle at points P, Q, S, and T, as shown. Also, $\angle AOB = \angle BOC = \angle COD = \angle DOA$. If $AO = 3$, determine the length of DS.



(b) Suppose that x satisfies $0 < x < \frac{\pi}{2}$ and $\cos\left(\frac{3}{2}\cos x\right) = \sin\left(\frac{3}{2}\sin x\right)$.

Determine all possible values of $\sin 2x$, expressing your answers in the form $\dfrac{a\pi^2 + b\pi + c}{d}$ where a, b, c, d are integers.

9. For positive integers a and b, define $f(a,b) = \dfrac{a}{b} + \dfrac{b}{a} + \dfrac{1}{ab}$.
For example, the value of $f(1,2)$ is 3.

(a) Determine the value of $f(2,5)$.

(b) Determine all positive integers a for which $f(a,a)$ is an integer.

(c) If a and b are positive integers and $f(a,b)$ is an integer, prove that $f(a,b)$ must be a multiple of 3.

(d) Determine four pairs of positive integers (a,b), with $2 < a < b$, for which $f(a,b)$ is an integer.

10. (a) Amir and Brigitte play a card game. Amir starts with a hand of 6 cards: 2 red, 2 yellow and 2 green. Brigitte starts with a hand of 4 cards: 2 purple and 2 white. Amir plays first. Amir and Brigitte alternate turns. On each turn, the current player chooses one of their own cards at random and places it on the table. The cards remain on the table for the rest of the game. A player wins and the game ends when they have placed two cards of the same colour on the table. Determine the probability that Amir wins the game.

(b) Carlos has 14 coins, numbered 1 to 14. Each coin has exactly one face called "heads". When flipped, coins $1, 2, 3, \ldots, 13, 14$ land heads with probabilities $h_1, h_2, h_3, \ldots, h_{13}, h_{14}$, respectively. When Carlos flips each of the 14 coins exactly once, the probability that an even number of coins land heads is exactly $\frac{1}{2}$. Must there be a k between 1 and 14, inclusive, for which $h_k = \frac{1}{2}$? Prove your answer.

(c) Serge and Lis each have a machine that prints a digit from 1 to 6. Serge's machine prints the digits $1, 2, 3, 4, 5, 6$ with probability p_1, p_2, p_3, p_4, p_5, p_6, respectively. Lis's machine prints the digits $1, 2, 3, 4, 5, 6$ with probability q_1, q_2, q_3, q_4, q_5, q_6, respectively. Each of the machines prints one digit. Let $S(i)$ be the probability that the sum of the two digits printed is i. If $S(2) = S(12) = \frac{1}{2}S(7)$ and $S(7) > 0$, prove that $p_1 = p_6$ and $q_1 = q_6$.

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For students...

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- Find your school's contest results

The CENTRE for EDUCATION
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Euclid Contest

Wednesday, April 11, 2018
(in North America and South America)

Thursday, April 12, 2018
(outside of North America and South America)

UNIVERSITY OF
WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) previously stored information such as formulas, programs, notes, etc., (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:

1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked (lightbulb), place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked (pencil), provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

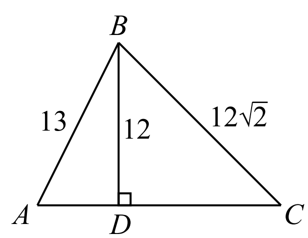
1. (a) If $x = 11$, what is the value of $x + (x+1) + (x+2) + (x+3)$?

   (b) If $\dfrac{a}{6} + \dfrac{6}{18} = 1$, what is the value of a?

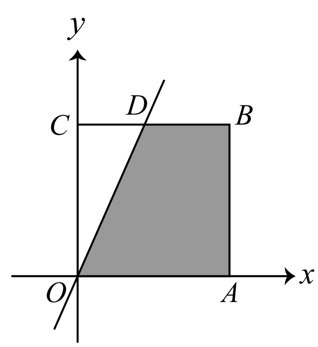
   (c) The total cost of one chocolate bar and two identical packs of gum is \$4.15. One chocolate bar costs \$1.00 more than one pack of gum. Determine the cost of one chocolate bar.

2. (a) A five-digit integer is made using each of the digits $1, 3, 5, 7, 9$. The integer is greater than 80 000 and less than 92 000. The units (ones) digit is 3. The hundreds and tens digits, in that order, form a two-digit integer that is divisible by 5. What is the five-digit integer?

   (b) In the diagram, point D is on AC so that BD is perpendicular to AC. Also, $AB = 13$, $BC = 12\sqrt{2}$ and $BD = 12$. What is the length of AC?



   (c) In the diagram, square $OABC$ has side length 6. The line with equation $y = 2x$ intersects CB at D. Determine the area of the shaded region.



3. (a) What is the value of $\left(\sqrt{4+\sqrt{4}}\right)^4$?

(b) There is exactly one pair (x, y) of positive integers for which $\sqrt{23-x} = 8 - y^2$. What is this pair (x, y)?

(c) The line with equation $y = mx + 2$ intersects the parabola with equation $y = ax^2 + 5x - 2$ at the points $P(1, 5)$ and Q. Determine

(i) the value of m,

(ii) the value of a, and

(iii) the coordinates of Q.

4. (a) The positive integers 34 and 80 have exactly two positive common divisors, namely 1 and 2. How many positive integers n with $1 \le n \le 30$ have the property that n and 80 have exactly two positive common divisors?

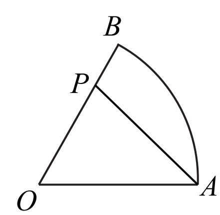
(b) A function f is defined so that

- $f(1) = 1$,
- if n is an even positive integer, then $f(n) = f\left(\frac{1}{2}n\right)$, and
- if n is an odd positive integer with $n > 1$, then $f(n) = f(n-1) + 1$.

For example, $f(34) = f(17)$ and $f(17) = f(16) + 1$.
Determine the value of $f(50)$.

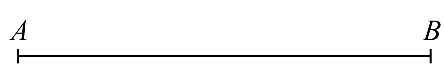
5. (a) The perimeter of equilateral $\triangle PQR$ is 12. The perimeter of regular hexagon $STUVWX$ is also 12. What is the ratio of the area of $\triangle PQR$ to the area of $STUVWX$?

(b) In the diagram, sector AOB is $\frac{1}{6}$ of an entire circle with radius $AO = BO = 18$. The sector is cut into two regions with a single straight cut through A and point P on OB. The areas of the two regions are equal. Determine the length of OP.



6. (a) For how many integers k with $0 < k < 18$ is $\dfrac{5\sin(10k^\circ) - 2}{\sin^2(10k^\circ)} \ge 2$?

(b) In the diagram, a straight, flat road joins A to B.



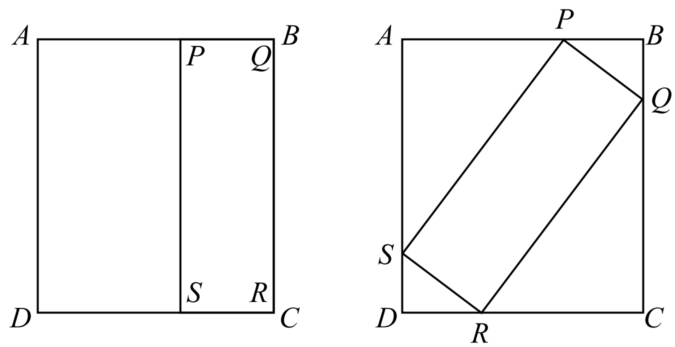
Karuna runs from A to B, turns around instantly, and runs back to A. Karuna runs at 6 m/s. Starting at the same time as Karuna, Jorge runs from B to A, turns around instantly, and runs back to B. Jorge runs from B to A at 5 m/s and from A to B at 7.5 m/s. The distance from A to B is 297 m and each runner takes exactly 99 s to run their route. Determine the two values of t for which Karuna and Jorge are at the same place on the road after running for t seconds.

7. (a) Eight people, including triplets Barry, Carrie and Mary, are going for a trip in four canoes. Each canoe seats two people. The eight people are to be randomly assigned to the four canoes in pairs. What is the probability that no two of Barry, Carrie and Mary will be in the same canoe?

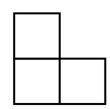
(b) Diagonal WY of square $WXYZ$ has slope 2. Determine the sum of the slopes of WX and XY.

8. (a) Determine all values of x such that $\log_{2x}(48\sqrt[3]{3}) = \log_{3x}(162\sqrt[3]{2})$.

(b) In the diagram, rectangle $PQRS$ is placed inside rectangle $ABCD$ in two different ways: first, with Q at B and R at C; second, with P on AB, Q on BC, R on CD, and S on DA.



If $AB = 718$ and $PQ = 250$, determine the length of BC.

9. An L-shaped triomino is composed of three unit squares, as shown: 

Suppose that H and W are positive integers. An $H \times W$ rectangle can be *tiled* if the rectangle can be completely covered with non-overlapping copies of this triomino (each of which can be rotated and/or translated) and the sum of the areas of these non-overlapping triominos equals the area of the rectangle (that is, no triomino is partly outside the rectangle). If such a rectangle can be tiled, a *tiling* is a specific configuration of triominos that tile the rectangle.

(a) Draw a tiling of a 3×8 rectangle.

(b) Determine, with justification, all integers W for which a $6 \times W$ rectangle can be tiled.

(c) Determine, with justification, all pairs (H, W) of integers with $H \geq 4$ and $W \geq 4$ for which an $H \times W$ rectangle can be tiled.

10. In an infinite array with two rows, the numbers in the top row are denoted $\ldots, A_{-2}, A_{-1}, A_0, A_1, A_2, \ldots$ and the numbers in the bottom row are denoted $\ldots, B_{-2}, B_{-1}, B_0, B_1, B_2, \ldots$. For each integer k, the entry A_k is directly above the entry B_k in the array, as shown:

$\ldots$	A_{-2}	A_{-1}	A_0	A_1	A_2	$\ldots$
$\ldots$	B_{-2}	B_{-1}	B_0	B_1	B_2	$\ldots$

For each integer k, A_k is the average of the entry to its left, the entry to its right, and the entry below it; similarly, each entry B_k is the average of the entry to its left, the entry to its right, and the entry above it.

(a) In one such array, $A_0 = A_1 = A_2 = 0$ and $A_3 = 1$.
Determine the value of A_4.
The maximum mark on this part is 2 marks.

(b) In another such array, we define $S_k = A_k + B_k$ for each integer k.
Prove that $S_{k+1} = 2S_k - S_{k-1}$ for each integer k.
The maximum mark on this part is 2 marks.

(c) Consider the following two statements about a third such array:

(P) If each entry is a positive integer, then all of the entries in the array are equal.

(Q) If each entry is a positive real number, then all of the entries in the array are equal.

Prove statement (Q).
The maximum mark on this part is 6 marks.

A complete proof of statement (Q) will earn the maximum of 6 marks for part (c), regardless of whether any attempt to prove (P) is made.
A complete proof of statement (P) will earn 2 of the 6 possible marks for part (c). In such a case, any further progress towards proving (Q) would be assessed for partial marks towards the remaining 4 marks.
Students who do not fully prove either (P) or (Q) will have their work assessed for partial marks.

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Euclid Contest

Thursday, April 6, 2017
(in North America and South America)

Friday, April 7, 2017
(outside of North America and South America)

UNIVERSITY OF
WATERLOO

Time: $2\frac{1}{2}$ hours

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A Note about Bubbling

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1. (a) There is one pair (a, b) of positive integers for which $5a + 3b = 19$. What are the values of a and b?

   (b) How many positive integers n satisfy $5 < 2^n < 2017$?

   (c) Jimmy bought 600 Euros at the rate of 1 Euro equals \$1.50. He then converted his 600 Euros back into dollars at the rate of \$1.00 equals 0.75 Euros. How many fewer dollars did Jimmy have after these two transactions than he had before these two transactions?

2. (a) What are all values of x for which $x \neq 0$ and $x \neq 1$ and $\dfrac{5}{x(x-1)} = \dfrac{1}{x} + \dfrac{1}{x-1}$?

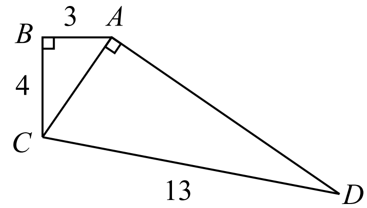
   (b) In a magic square, the numbers in each row, the numbers in each column, and the numbers on each diagonal have the same sum. In the magic square shown, what are the values of a, b and c?

0	20	a
c	4	
	-12	b

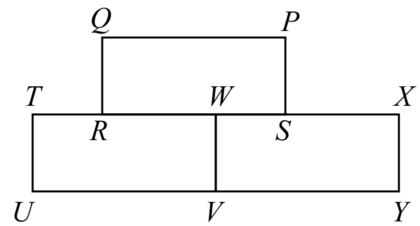
   (c) (i) For what positive integer n is $100^2 - n^2 = 9559$?

   (ii) Determine one pair (a, b) of positive integers for which $a > 1$ and $b > 1$ and $ab = 9559$.

3. (a) In the diagram, $\triangle ABC$ is right-angled at B and $\triangle ACD$ is right-angled at A. Also, $AB = 3$, $BC = 4$, and $CD = 13$. What is the area of quadrilateral $ABCD$?



(b) Three identical rectangles $PQRS$, $WTUV$ and $XWVY$ are arranged, as shown, so that RS lies along TX. The perimeter of each of the three rectangles is 21 cm. What is the perimeter of the whole shape?



(c) One of the faces of a rectangular prism has area 27 cm^2. Another face has area 32 cm^2. If the volume of the prism is 144 cm^3, determine the surface area of the prism in cm^2.

4. (a) The equations $y = a(x-2)(x+4)$ and $y = 2(x-h)^2 + k$ represent the same parabola. What are the values of a, h and k?

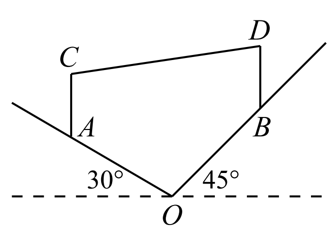
(b) In an arithmetic sequence with 5 terms, the sum of the squares of the first 3 terms equals the sum of the squares of the last 2 terms. If the first term is 5, determine all possible values of the fifth term.

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9, 11$ is an arithmetic sequence with five terms.)

5. (a) Dan was born in a year between 1300 and 1400. Steve was born in a year between 1400 and 1500. Each was born on April 6 in a year that is a perfect square. Each lived for 110 years. In what year while they were both alive were their ages both perfect squares on April 7?

(b) Determine all values of k for which the points $A(1,2)$, $B(11,2)$ and $C(k,6)$ form a right-angled triangle.

6. (a) The diagram shows two hills that meet at O. One hill makes a 30° angle with the horizontal and the other hill makes a 45° angle with the horizontal. Points A and B are on the hills so that $OA = OB = 20$ m. Vertical poles BD and AC are connected by a straight cable CD. If $AC = 6$ m, what is the length of BD for which CD is as short as possible?



(b) If $\cos\theta = \tan\theta$, determine all possible values of $\sin\theta$, giving your answer(s) as simplified exact numbers.

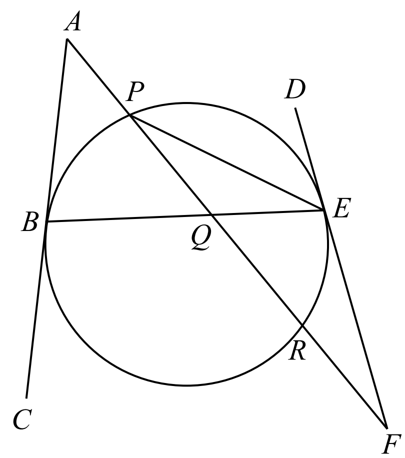
7\. (a) Linh is driving at 60 km/h on a long straight highway parallel to a train track. Every 10 minutes, she is passed by a train travelling in the same direction as she is. These trains depart from the station behind her every 3 minutes and all travel at the same constant speed. What is the constant speed of the trains, in km/h?

(b) Determine all pairs (a, b) of real numbers that satisfy the following system of equations:

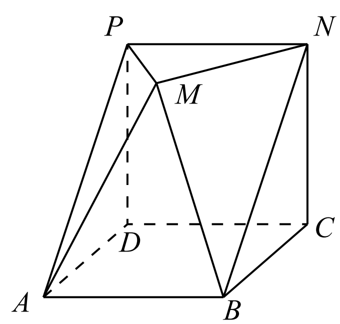
$$\sqrt{a} + \sqrt{b} = 8$$
$$\log_{10} a + \log_{10} b = 2$$

Give your answer(s) as pairs of simplified exact numbers.

8\. (a) In the diagram, line segments AC and DF are tangent to the circle at B and E, respectively. Also, AF intersects the circle at P and R, and intersects BE at Q, as shown. If $\angle CAF = 35^\circ$, $\angle DFA = 30^\circ$, and $\angle FPE = 25^\circ$, determine the measure of $\angle PEQ$.



(b) In the diagram, $ABCD$ and $PNCD$ are squares of side length 2, and $PNCD$ is perpendicular to $ABCD$. Point M is chosen on the same side of $PNCD$ as AB so that $\triangle PMN$ is parallel to $ABCD$, so that $\angle PMN = 90^\circ$, and so that $PM = MN$. Determine the volume of the convex solid $ABCDPMN$.



9. A *permutation* of a list of numbers is an ordered arrangement of the numbers in that list. For example, $3, 2, 4, 1, 6, 5$ is a permutation of $1, 2, 3, 4, 5, 6$. We can write this permutation as $a_1, a_2, a_3, a_4, a_5, a_6$, where $a_1 = 3, a_2 = 2, a_3 = 4, a_4 = 1, a_5 = 6$, and $a_6 = 5$.

(a) Determine the average value of

$$|a_1 - a_2| + |a_3 - a_4|$$

over all permutations a_1, a_2, a_3, a_4 of $1, 2, 3, 4$.

(b) Determine the average value of

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7$$

over all permutations $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ of $1, 2, 3, 4, 5, 6, 7$.

(c) Determine the average value of

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_{197} - a_{198}| + |a_{199} - a_{200}| \qquad (*)$$

over all permutations $a_1, a_2, a_3, ..., a_{199}, a_{200}$ of $1, 2, 3, 4, \ldots, 199, 200$. (The sum labelled $(*)$ contains 100 terms of the form $|a_{2k-1} - a_{2k}|$.)

10. Consider a set S that contains $m \geq 4$ elements, each of which is a positive integer and no two of which are equal. We call S *boring* if it contains four distinct integers a, b, c, d such that $a + b = c + d$. We call S *exciting* if it is not boring. For example, $\{2, 4, 6, 8, 10\}$ is boring since $4 + 8 = 2 + 10$. Also, $\{1, 5, 10, 25, 50\}$ is exciting.

(a) Find an exciting subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that contains exactly 5 elements.

(b) Prove that, if S is an exciting set of $m \geq 4$ positive integers, then S contains an integer greater than or equal to $\dfrac{m^2 - m}{4}$.

(c) Define $\text{rem}(a, b)$ to be the remainder when the positive integer a is divided by the positive integer b. For example, $\text{rem}(10, 7) = 3$, $\text{rem}(20, 5) = 0$, and $\text{rem}(3, 4) = 3$.

Let n be a positive integer with $n \geq 10$. For each positive integer k with $1 \leq k \leq n$, define $x_k = 2n \cdot \text{rem}(k^2, n) + k$. Determine, with proof, all positive integers $n \geq 10$ for which the set $\{x_1, x_2, \ldots, x_{n-1}, x_n\}$ of n integers is exciting.

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For students...

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Euclid Contest

Tuesday, April 12, 2016
(in North America and South America)

Wednesday, April 13, 2016
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

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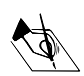
2. **FULL SOLUTION** parts indicated by 
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   - **must be written in the appropriate location** in the answer booklet
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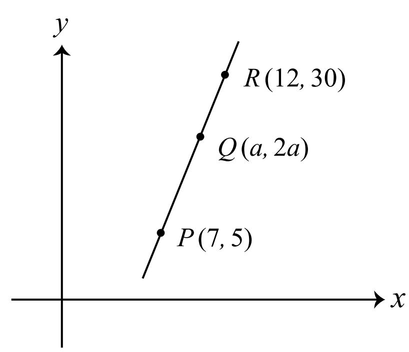
A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

1. (a) What is the average of the integers $5, 15, 25, 35, 45, 55$?

   (b) If $x^2 = 2016$, what is the value of $(x + 2)(x - 2)$?

   (c) In the diagram, points $P(7, 5)$, $Q(a, 2a)$, and $R(12, 30)$ lie on a straight line. Determine the value of a.

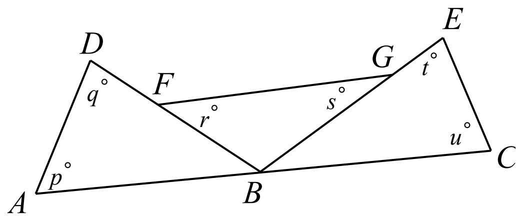


2. (a) What are all values of n for which $\dfrac{n}{9} = \dfrac{25}{n}$?

   (b) What are all values of x for which $(x - 3)(x - 2) = 6$?

   (c) At Willard's Grocery Store, the cost of 2 apples is the same as the cost of 3 bananas. Ross buys 6 apples and 12 bananas for a total cost of \$6.30. Determine the cost of 1 apple.

3. (a) In the diagram, point B is on AC, point F is on DB, and point G is on EB.



What is the value of $p + q + r + s + t + u$?

(b) Let n be the integer equal to $10^{20} - 20$. What is the sum of the digits of n?

(c) A parabola intersects the x-axis at $P(2, 0)$ and $Q(8, 0)$. The vertex of the parabola is at V, which is below the x-axis. If the area of $\triangle VPQ$ is 12, determine the coordinates of V.

4. (a) Determine all angles θ with $0^\circ \le \theta \le 180^\circ$ and $\sin^2\theta + 2\cos^2\theta = \frac{7}{4}$.

(b) The sum of the radii of two circles is 10 cm. The circumference of the larger circle is 3 cm greater than the circumference of the smaller circle. Determine the difference between the area of the larger circle and the area of the smaller circle.

5. (a) Charlotte's Convenience Centre buys a calculator for $\$p$ (where $p > 0$), raises its price by $n\%$, then reduces this new price by 20%. If the final price is 20% higher than $\$p$, what is the value of n?

(b) A function f is defined so that if n is an odd integer, then $f(n) = n - 1$ and if n is an even integer, then $f(n) = n^2 - 1$. For example, if $n = 15$, then $f(n) = 14$ and if $n = -6$, then $f(n) = 35$, since 15 is an odd integer and -6 is an even integer. Determine all integers n for which $f(f(n)) = 3$.

6. (a) What is the smallest positive integer x for which $\dfrac{1}{32} = \dfrac{x}{10^y}$ for some positive integer y?

(b) Determine all possible values for the area of a right-angled triangle with one side length equal to 60 and with the property that its side lengths form an arithmetic sequence.

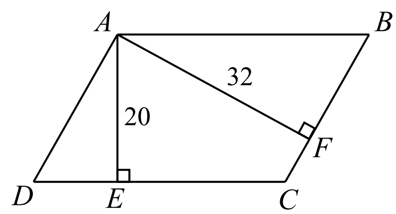
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.)

7. (a) Amrita and Zhang cross a lake in a straight line with the help of a one-seat kayak. Each can paddle the kayak at 7 km/h and swim at 2 km/h. They start from the same point at the same time with Amrita paddling and Zhang swimming. After a while, Amrita stops the kayak and immediately starts swimming. Upon reaching the kayak (which has not moved since Amrita started swimming), Zhang gets in and immediately starts paddling. They arrive on the far side of the lake at the same time, 90 minutes after they began. Determine the amount of time during these 90 minutes that the kayak was not being paddled.

(b) Determine all pairs (x, y) of real numbers that satisfy the system of equations

$$\begin{aligned} x\left(\tfrac{1}{2} + y - 2x^2\right) &= 0 \\ y\left(\tfrac{5}{2} + x - y\right) &= 0 \end{aligned}$$

8. (a) In the diagram, $ABCD$ is a parallelogram. Point E is on DC with AE perpendicular to DC, and point F is on CB with AF perpendicular to CB. If $AE = 20$, $AF = 32$, and $\cos(\angle EAF) = \frac{1}{3}$, determine the exact value of the area of quadrilateral $AECF$.

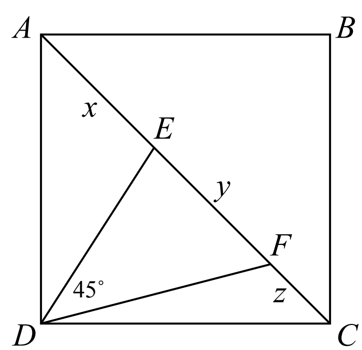


(b) Determine all real numbers $x > 0$ for which

$$\log_4 x - \log_x 16 = \tfrac{7}{6} - \log_x 8$$

9. (a) The string $AAABBBAABB$ is a string of ten letters, each of which is A or B, that does not include the consecutive letters $ABBA$.
The string $AAABBAAABB$ is a string of ten letters, each of which is A or B, that does include the consecutive letters $ABBA$.
Determine, with justification, the total number of strings of ten letters, each of which is A or B, that do not include the consecutive letters $ABBA$.

(b) In the diagram, $ABCD$ is a square. Points E and F are chosen on AC so that $\angle EDF = 45^\circ$. If $AE = x$, $EF = y$, and $FC = z$, prove that $y^2 = x^2 + z^2$.



10. Let k be a positive integer with $k \geq 2$. Two bags each contain k balls, labelled with the positive integers from 1 to k. André removes one ball from each bag. (In each bag, each ball is equally likely to be chosen.) Define $P(k)$ to be the probability that the product of the numbers on the two balls that he chooses is divisible by k.

(a) Calculate $P(10)$.

(b) Determine, with justification, a polynomial $f(n)$ for which

- $P(n) \geq \dfrac{f(n)}{n^2}$ for all positive integers n with $n \geq 2$, and
- $P(n) = \dfrac{f(n)}{n^2}$ for infinitely many positive integers n with $n \geq 2$.

(A *polynomial* $f(x)$ is an algebraic expression of the form $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ for some integer $m \geq 0$ and for some real numbers $a_m, a_{m-1}, \ldots, a_1, a_0$.)

(c) Prove there exists a positive integer m for which $P(m) > \dfrac{2016}{m}$.

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Euclid Contest

Wednesday, April 15, 2015
(in North America and South America)

Thursday, April 16, 2015
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

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A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

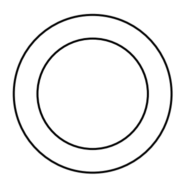
1. (a) What is value of $\dfrac{10^2 - 9^2}{10 + 9}$?

   (b) If $\dfrac{x+1}{x+4} = 4$, what is the value of $3x + 8$?

   (c) If $f(x) = 2x - 1$, determine the value of $(f(3))^2 + 2(f(3)) + 1$.

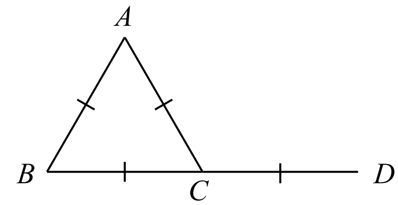
2. (a) If $\sqrt{a} + \sqrt{a} = 20$, what is the value of a?

   (b) Two circles have the same centre. The radius of the smaller circle is 1. The area of the region between the circles is equal to the area of the smaller circle. What is the radius of the larger circle?



   (c) There were 30 students in Dr. Brown's class. The average mark of the students in the class was 80. After two students dropped the class, the average mark of the remaining students was 82. Determine the average mark of the two students who dropped the class.

3. (a) In the diagram, $BD = 4$ and point C is the midpoint of BD. If point A is placed so that $\triangle ABC$ is equilateral, what is the length of AD?

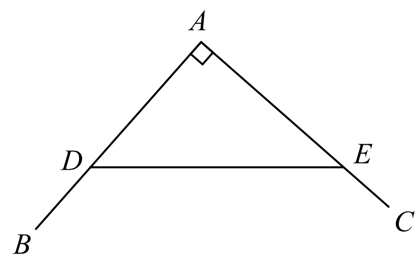


(b) $\triangle MNP$ has vertices $M(1, 4)$, $N(5, 3)$, and $P(5, c)$. Determine the sum of the two values of c for which the area of $\triangle MNP$ is 14.

4. (a) What are the x-intercepts and the y-intercept of the graph with equation $y = (x-1)(x-2)(x-3) - (x-2)(x-3)(x-4)$?

(b) The graphs of the equations $y = x^3 - x^2 + 3x - 4$ and $y = ax^2 - x - 4$ intersect at exactly two points. Determine all possible values of a.

5. (a) In the diagram, $\angle CAB = 90°$. Point D is on AB and point E is on AC so that $AB = AC = DE$, $DB = 9$, and $EC = 8$. Determine the length of DE.

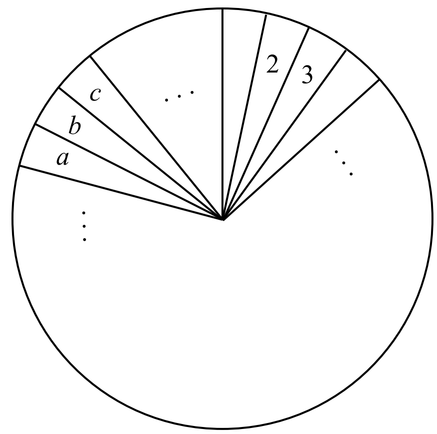


(b) Ellie has two lists, each consisting of 6 consecutive positive integers. The smallest integer in the first list is a, the smallest integer in the second list is b, and $a < b$. She makes a third list which consists of the 36 integers formed by multiplying each number from the first list with each number from the second list. (This third list may include some repeated numbers.) If

- the integer 49 appears in the third list,
- there is no number in the third list that is a multiple of 64, and
- there is at least one number in the third list that is larger than 75,

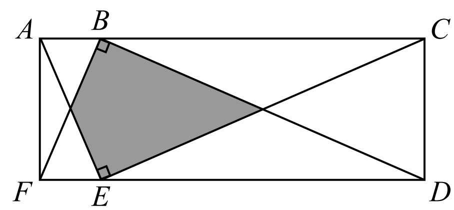
determine all possible pairs (a, b).

6. (a) A circular disc is divided into 36 sectors. A number is written in each sector. When three consecutive sectors contain a, b and c in that order, then $b = ac$. If the number 2 is placed in one of the sectors and the number 3 is placed in one of the adjacent sectors, as shown, what is the sum of the 36 numbers on the disc?



(b) Determine all values of x for which $0 < \dfrac{x^2 - 11}{x+1} < 7$.

7. (a) In the diagram, $ACDF$ is a rectangle with $AC = 200$ and $CD = 50$. Also, $\triangle FBD$ and $\triangle AEC$ are congruent triangles which are right-angled at B and E, respectively. What is the area of the shaded region?

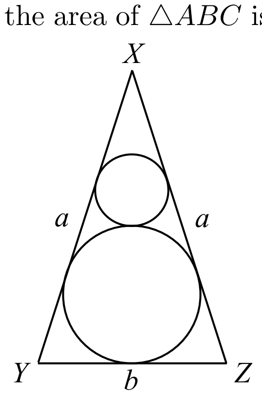


(b) The numbers $a_1, a_2, a_3, \ldots$ form an arithmetic sequence with $a_1 \neq a_2$. The three numbers a_1, a_2, a_6 form a geometric sequence in that order. Determine all possible positive integers k for which the three numbers a_1, a_4, a_k also form a geometric sequence in that order.

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.
A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

8. (a) For some positive integers k, the parabola with equation $y = \dfrac{x^2}{k} - 5$ intersects the circle with equation $x^2 + y^2 = 25$ at exactly three distinct points A, B and C. Determine all such positive integers k for which the area of $\triangle ABC$ is an integer.

(b) In the diagram, $\triangle XYZ$ is isosceles with $XY = XZ = a$ and $YZ = b$ where $b < 2a$. A larger circle of radius R is inscribed in the triangle (that is, the circle is drawn so that it touches all three sides of the triangle). A smaller circle of radius r is drawn so that it touches XY, XZ and the larger circle. Determine an expression for $\dfrac{R}{r}$ in terms of a and b.



9. Consider the following system of equations in which all logarithms have base 10:

$$
\begin{aligned}
(\log x)(\log y) - 3\log 5y - \log 8x &= a \\
(\log y)(\log z) - 4\log 5y - \log 16z &= b \\
(\log z)(\log x) - 4\log 8x - 3\log 625z &= c
\end{aligned}
$$

(a) If $a = -4$, $b = 4$, and $c = -18$, solve the system of equations.

(b) Determine all triples (a, b, c) of real numbers for which the system of equations has an infinite number of solutions (x, y, z).

10. For each positive integer $n \geq 1$, let C_n be the set containing the n smallest positive integers; that is, $C_n = \{1, 2, \ldots, n-1, n\}$. For example, $C_4 = \{1, 2, 3, 4\}$. We call a set, F, of subsets of C_n a *Furoni family of* C_n if no element of F is a subset of another element of F.

(a) Consider $A = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$. Note that A is a Furoni family of C_4. Determine the two Furoni families of C_4 that contain all of the elements of A and to which no other subsets of C_4 can be added to form a new (larger) Furoni family.

(b) Suppose that n is a positive integer and that F is a Furoni family of C_n. For each non-negative integer k, define a_k to be the number of elements of F that contain exactly k integers. Prove that

$$\frac{a_0}{\binom{n}{0}} + \frac{a_1}{\binom{n}{1}} + \frac{a_2}{\binom{n}{2}} + \cdots + \frac{a_{n-1}}{\binom{n}{n-1}} + \frac{a_n}{\binom{n}{n}} \leq 1$$

(The sum on the left side includes $n + 1$ terms.)

(Note: If n is a positive integer and k is an integer with $0 \leq k \leq n$, then $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ is the number of subsets of C_n that contain exactly k integers, where $0! = 1$ and, if m is a positive integer, $m!$ represents the product of the integers from 1 to m, inclusive.)

(c) For each positive integer n, determine, with proof, the number of elements in the largest Furoni family of C_n (that is, the number of elements in the Furoni family that contains the maximum possible number of subsets of C_n).

2015 Euclid Contest (English)

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Euclid Contest

Tuesday, April 15, 2014
(in North America and South America)

Wednesday, April 16, 2014
(outside of North America and South America)

UNIVERSITY OF WATERLOO   WATERLOO MATHEMATICS   Deloitte.

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Time: $2\frac{1}{2}$ hours
Calculators are permitted, provided they are non-programmable and without graphic displays.

Number of questions: 10
Each question is worth 10 marks

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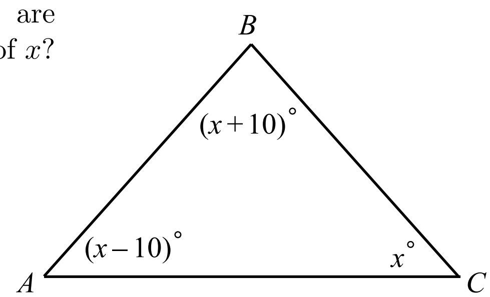
A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth, grade, and sex, on the Student Information Form, and that you have answered the question about eligibility.

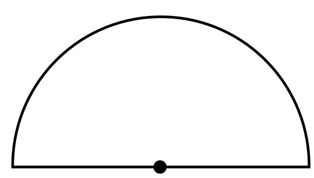
1. (a) What is the value of $\dfrac{\sqrt{16}+\sqrt{9}}{\sqrt{16+9}}$?

   (b) In the diagram, the angles of $\triangle ABC$ are shown in terms of x. What is the value of x?



   (c) Lisa earns two times as much per hour as Bart. Lisa works 6 hours and Bart works 4 hours. They earn $200 in total. How much does Lisa earn per hour?

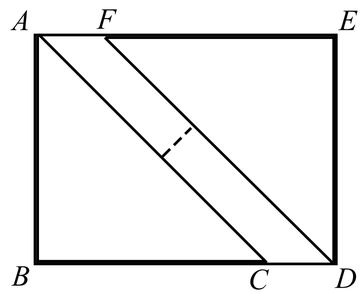
2. (a) The semi-circular region shown has radius 10. What is the perimeter of the region?



   (b) The parabola with equation $y = 10(x+2)(x-5)$ intersects the x-axis at points P and Q. What is the length of line segment PQ?

   (c) The line with equation $y = 2x$ intersects the line segment joining $C(0,60)$ and $D(30,0)$ at the point E. Determine the coordinates of E.

3. (a) Jimmy is baking two large identical triangular cookies, $\triangle ABC$ and $\triangle DEF$. Each cookie is in the shape of an isosceles right-angled triangle. The length of the shorter sides of each of these triangles is 20 cm. He puts the cookies on a rectangular baking tray so that A, B, D, and E are at the vertices of the rectangle, as shown. If the distance between parallel sides AC and DF is 4 cm, what is the width BD of the tray?



(b) Determine all values of x for which $\dfrac{x^2+x+4}{2x+1}=\dfrac{4}{x}$.

4. (a) Determine the number of positive divisors of 900, including 1 and 900, that are perfect squares. (A *positive divisor* of 900 is a positive integer that divides exactly into 900.)

(b) Points $A(k,3)$, $B(3,1)$ and $C(6,k)$ form an isosceles triangle. If $\angle ABC = \angle ACB$, determine all possible values of k.

5. (a) A chemist has three bottles, each containing a mixture of acid and water:

- bottle A contains 40 g of which 10% is acid,
- bottle B contains 50 g of which 20% is acid, and
- bottle C contains 50 g of which 30% is acid.

She uses some of the mixture from each of the bottles to create a mixture with mass 60 g of which 25% is acid. Then she mixes the remaining contents of the bottles to create a new mixture. What percentage of the new mixture is acid?

(b) Suppose that x and y are real numbers with $3x+4y=10$. Determine the minimum possible value of x^2+16y^2.

6. (a) A bag contains 40 balls, each of which is black or gold. Feridun reaches into the bag and randomly removes two balls. Each ball in the bag is equally likely to be removed. If the probability that two gold balls are removed is $\frac{5}{12}$, how many of the 40 balls are gold?

(b) The geometric sequence with n terms $t_1, t_2, \ldots, t_{n-1}, t_n$ has $t_1 t_n = 3$. Also, the product of all n terms equals 59 049 (that is, $t_1 t_2 \cdots t_{n-1} t_n = 59\,049$). Determine the value of n.

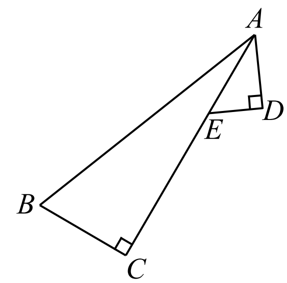
(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

7. (a) If $\dfrac{(x-2013)(y-2014)}{(x-2013)^2+(y-2014)^2} = -\dfrac{1}{2}$, what is the value of $x+y$?

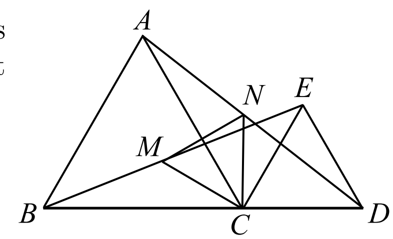
(b) Determine all real numbers x for which

$$(\log_{10} x)^{\log_{10}(\log_{10} x)} = 10\,000$$

8. (a) In the diagram, $\angle ACB = \angle ADE = 90^\circ$. If $AB = 75$, $BC = 21$, $AD = 20$, and $CE = 47$, determine the exact length of BD.



(b) In the diagram, C lies on BD. Also, $\triangle ABC$ and $\triangle ECD$ are equilateral triangles. If M is the midpoint of BE and N is the midpoint of AD, prove that $\triangle MNC$ is equilateral.



9. (a) Without using a calculator, determine positive integers m and n for which

$$\sin^6 1^\circ + \sin^6 2^\circ + \sin^6 3^\circ + \cdots + \sin^6 87^\circ + \sin^6 88^\circ + \sin^6 89^\circ = \frac{m}{n}$$

(The sum on the left side of the equation consists of 89 terms of the form $\sin^6 x^\circ$, where x takes each positive integer value from 1 to 89.)

(b) Let $f(n)$ be the number of positive integers that have exactly n digits and whose digits have a sum of 5. Determine, with proof, how many of the 2014 integers $f(1), f(2), \ldots, f(2014)$ have a units digit of 1.

10. Fiona plays a game with jelly beans on the number line. Initially, she has N jelly beans, all at position 0. On each turn, she must choose one of the following moves:

- Type 1: She removes two jelly beans from position 0, eats one, and puts the other at position 1.
- Type i, where i is an integer with $i \geq 2$: She removes one jelly bean from position $i-2$ and one jelly bean from position $i-1$, eats one, and puts the other at position i.

The positions of the jelly beans when no more moves are possible is called the *final state*. Once a final state is reached, Fiona is said to have won the game if there are at most three jelly beans remaining, each at a distinct position and no two at consecutive integer positions. For example, if $N = 7$, Fiona wins the game with the sequence of moves

Type 1, Type 1, Type 2, Type 1, Type 3

which leaves jelly beans at positions 1 and 3. A different sequence of moves starting with $N = 7$ might not win the game.

(a) Determine an integer N for which it is possible to win the game with one jelly bean left at position 5 and no jelly beans left at any other position.

(b) Suppose that Fiona starts the game with a fixed unknown positive integer N. Prove that if Fiona can win the game, then there is only one possible final state.

(c) Determine, with justification, the closest positive integer N to 2014 for which Fiona can win the game.

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For students...

Thank you for writing the 2014 Euclid Contest!
In 2013, more than 17 000 students from around the world registered to write the Euclid Contest.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2014 Canadian Senior Mathematics Contest, which will be written in November 2014.

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Euclid Contest

Wednesday, April 17, 2013
(in North America and South America)

Thursday, April 18, 2013
(outside of North America and South America)

Do not open this booklet until instructed to do so.

Time: $2\frac{1}{2}$ hours
Calculators are permitted, provided they are non-programmable and without graphic displays.

Number of questions: 10
Each question is worth 10 marks

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express calculations and answers as exact numbers such as $\pi + 1$ and $\sqrt{2}$, etc., rather than as $4.14\ldots$ or $1.41\ldots$, except where otherwise indicated.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, http://www.cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

TIPS:	1.	Please read the instructions on the front cover of this booklet.
	2.	Write all answers in the answer booklet provided.
	3.	For questions marked (lightbulb), place your answer in the appropriate box in the answer booklet and **show your work**.
	4.	For questions marked (pencil), provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
	5.	Diagrams are *not* drawn to scale. They are intended as aids only.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth, grade, and sex, on the Student Information Form, and that you have answered the question about eligibility.

1. (a) What is the smallest positive integer x for which $\sqrt{113 + x}$ is an integer?

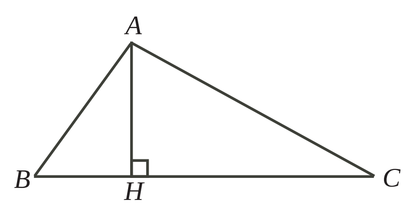
   (b) The average of 3 and 11 is a. The average of a and b is 11. What is the value of b?

   (c) Charlie is 30 years older than his daughter Bella. Charlie is also six times as old as Bella. Determine Charlie's age.

2. (a) If $\dfrac{21}{x} = \dfrac{7}{y}$ with $x \neq 0$ and $y \neq 0$, what is the value of $\dfrac{x}{y}$?

   (b) For which positive integer n are both $\dfrac{1}{n+1} < 0.2013$ and $0.2013 < \dfrac{1}{n}$ true?

   (c) In the diagram, H is on side BC of $\triangle ABC$ so that AH is perpendicular to BC. Also, $AB = 10$, $AH = 8$, and the area of $\triangle ABC$ is 84. Determine the perimeter of $\triangle ABC$.



3. (a) In the Fibonacci sequence, $1, 1, 2, 3, 5, \ldots$, each term after the second is the sum of the previous two terms. How many of the first 100 terms of the Fibonacci sequence are odd?

   (b) In an arithmetic sequence, the sum of the first and third terms is 6 and the sum of the second and fourth terms is 20. Determine the tenth term in the sequence.

   (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.)

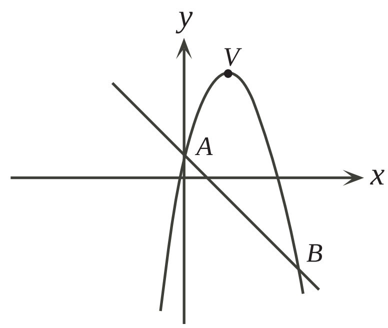
4. (a) How many positive integers less than 1000 have only odd digits?

(b) Determine all ordered pairs (a, b) that satisfy the following system of equations.

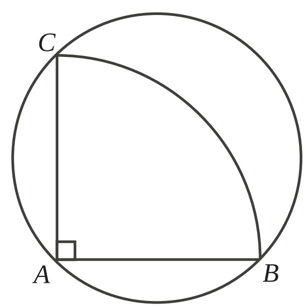
$$\begin{aligned} a + b &= 16 \\ \frac{4}{7} &= \frac{1}{a} + \frac{1}{b} \end{aligned}$$

5. (a) Tanner has two identical dice. Each die has six faces which are numbered 2, 3, 5, 7, 11, 13. When Tanner rolls the two dice, what is the probability that the sum of the numbers on the top faces is a prime number?

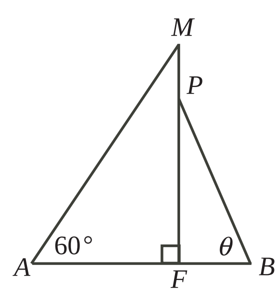
(b) In the diagram, V is the vertex of the parabola with equation $y = -x^2 + 4x + 1$. Also, A and B are the points of intersection of the parabola and the line with equation $y = -x + 1$. Determine the value of $AV^2 + BV^2 - AB^2$.



6. (a) In the diagram, ABC is a quarter of a circular pizza with centre A and radius 20 cm. The piece of pizza is placed on a circular pan with A, B and C touching the circumference of the pan, as shown. What fraction of the pan is covered by the piece of pizza?



(b) The deck AB of a sailboat is 8 m long. Rope extends at an angle of 60° from A to the top (M) of the mast of the boat. More rope extends at an angle of θ from B to a point P that is 2 m below M, as shown. Determine the height MF of the mast, in terms of θ.



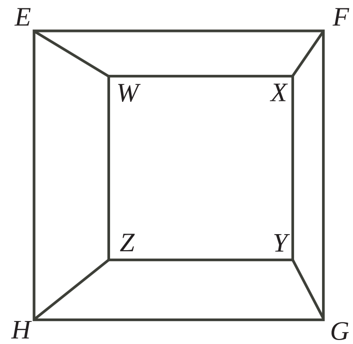
7. (a) If $\dfrac{1}{\cos x} - \tan x = 3$, what is the numerical value of $\sin x$?

(b) Determine all linear functions $f(x) = ax + b$ such that if $g(x) = f^{-1}(x)$ for all values of x, then $f(x) - g(x) = 44$ for all values of x. (Note: f^{-1} is the inverse function of f.)

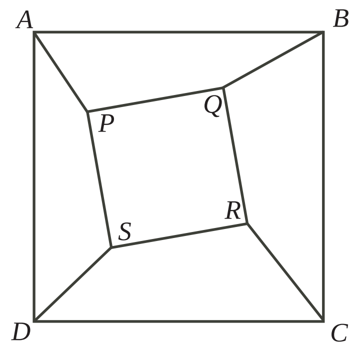
8\. (a) Determine all pairs (a, b) of positive integers for which $a^3 + 2ab = 2013$.

(b) Determine all real values of x for which $\log_2(2^{x-1} + 3^{x+1}) = 2x - \log_2(3^x)$.

9\. (a) Square $WXYZ$ has side length 6 and is drawn, as shown, completely inside a larger square $EFGH$ with side length 10, so that the squares do not touch and so that WX is parallel to EF. Prove that the sum of the areas of trapezoid $EFXW$ and trapezoid $GHZY$ does not depend on the position of $WXYZ$ inside $EFGH$.



(b) A large square $ABCD$ is drawn, with a second smaller square $PQRS$ completely inside it so that the squares do not touch. Line segments AP, BQ, CR, and DS are drawn, dividing the region between the squares into four non-overlapping convex quadrilaterals, as shown. If the sides of $PQRS$ are *not* parallel to the sides of $ABCD$, prove that the sum of the areas of quadrilaterals $APSD$ and $BCRQ$ equals the sum of the areas of quadrilaterals $ABQP$ and $CDSR$. (Note: A convex quadrilateral is a quadrilateral in which the measure of each of the four interior angles is less than $180°$.)



10\. A *multiplicative partition* of a positive integer $n \geq 2$ is a way of writing n as a product of one or more integers, each greater than 1. Note that we consider a positive integer to be a multiplicative partition of itself. Also, the order of the factors in a partition does not matter; for example, $2 \times 3 \times 5$ and $2 \times 5 \times 3$ are considered to be the same partition of 30. For each positive integer $n \geq 2$, define $P(n)$ to be the number of multiplicative partitions of n. We also define $P(1) = 1$. Note that $P(40) = 7$, since the multiplicative partitions of 40 are 40, 2×20, 4×10, 5×8, $2 \times 2 \times 10$, $2 \times 4 \times 5$, and $2 \times 2 \times 2 \times 5$.

(a) Determine the value of $P(64)$.

(b) Determine the value of $P(1000)$.

(c) Determine, with proof, a sequence of integers $a_0, a_1, a_2, a_3, \ldots$ with the property that

$$P(4 \times 5^m) = a_0 P(2^m) + a_1 P(2^{m-1}) + a_2 P(2^{m-2}) + \cdots + a_{m-1} P(2^1) + a_m P(2^0)$$

for every positive integer m.

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Euclid Contest

Wednesday, April 11, 2012
(in North America and South America)

Thursday, April 12, 2012
(outside of North America and South America)

UNIVERSITY OF WATERLOO

WATERLOO MATHEMATICS

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Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

Number of questions: 10
Each question is worth 10 marks

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided

2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express calculations and answers as exact numbers such as $\pi + 1$ and $\sqrt{2}$, etc., rather than as $4.14\ldots$ or $1.41\ldots$, except where otherwise indicated.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published in the Euclid Results on our Web site, http://www.cemc.uwaterloo.ca.

TIPS:		
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	2.	Write all answers in the answer booklet provided.
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A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth, grade, and sex, on the Student Information Form, and that you have answered the question about eligibility.

1. (a) John buys 10 bags of apples, each of which contains 20 apples. If he eats 8 apples a day, how many days will it take him to eat the 10 bags of apples?

   (b) Determine the value of

   $$\sin(0^\circ) + \sin(60^\circ) + \sin(120^\circ) + \sin(180^\circ) + \sin(240^\circ) + \sin(300^\circ) + \sin(360^\circ)$$

   (c) A set of integers has a sum of 420, and an average of 60. If one of the integers in the set is 120, what is average of the remaining integers in the set?

2. (a) If $ax + ay = 4$ and $x + y = 12$, what is the value of a?

   (b) If the lines with equations $4x + 6y = 5$ and $6x + ky = 3$ are parallel, what is the value of k?

   (c) Determine all pairs (x, y) that satisfy the system of equations

   $$\begin{aligned} x + y &= 0 \\ x^2 - y &= 2 \end{aligned}$$

3. (a) A 200 g solution consists of water and salt. 25% of the total mass of the solution is salt. How many grams of water need to be added in order to change the solution so that it is 10% salt by mass?

   (b) The correct formula for converting a Celsius temperature (C) to a Fahrenheit temperature (F) is given by $F = \frac{9}{5}C + 32$.
   To approximate the Fahrenheit temperature, Gordie doubles C and then adds 30 to get f.
   If $f < F$, the error in the approximation is $F - f$; otherwise, the error in the approximation is $f - F$. (For example, if $F = 68$ and $f = 70$, the error in the approximation is $f - F = 2$.)
   Determine the largest possible error in the approximation that Gordie would make when converting Celsius temperatures C with $-20 \le C \le 35$.

4. (a) The horizontal line $y = k$ intersects the parabola with equation $y = 2(x-3)(x-5)$ at points A and B. If the length of line segment AB is 6, what is the value of k?

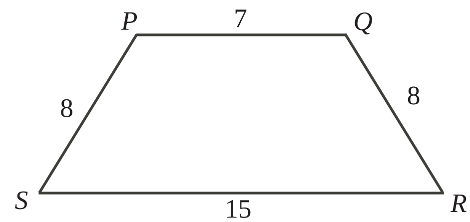
(b) Determine three pairs (a, b) of positive integers for which

$$(3a + 6a + 9a + 12a + 15a) + (6b + 12b + 18b + 24b + 30b)$$

is a perfect square.

5. (a) Triangle ABC has vertices $A(0, 5)$, $B(3, 0)$ and $C(8, 3)$. Determine the measure of $\angle ACB$.

(b) In the diagram, $PQRS$ is an isosceles trapezoid with $PQ = 7$, $PS = QR = 8$, and $SR = 15$. Determine the length of the diagonal PR.

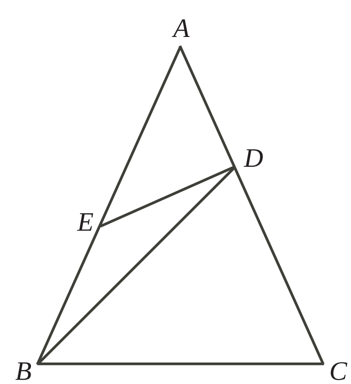


6. (a) Blaise and Pierre will play 6 games of squash. Since they are equally skilled, each is equally likely to win any given game. (In squash, there are no ties.) The probability that each of them will win 3 of the 6 games is $\frac{5}{16}$. What is the probability that Blaise will win more games than Pierre?

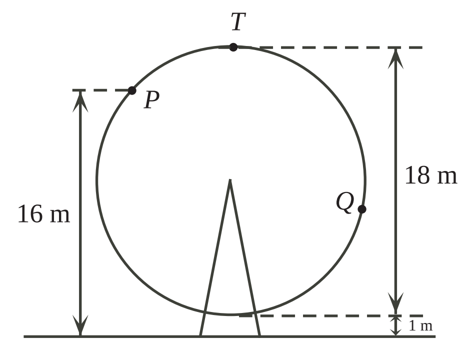
(b) Determine all real values of x for which

$$3^{x+2} + 2^{x+2} + 2^x = 2^{x+5} + 3^x$$

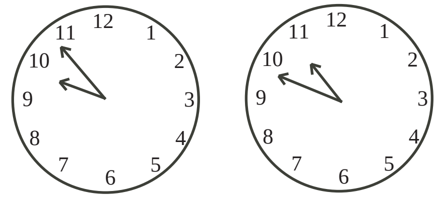
7. (a) In the diagram, $\triangle ABC$ has $AB = AC$ and $\angle BAC < 60°$. Point D is on AC with $BC = BD$. Point E is on AB with $BE = ED$. If $\angle BAC = \theta$, determine $\angle BED$ in terms of θ.



(b) In the diagram, the ferris wheel has a diameter of 18 m and rotates at a constant rate. When Kolapo rides the ferris wheel and is at its lowest point, he is 1 m above the ground. When Kolapo is at point P that is 16 m above the ground and is rising, it takes him 4 seconds to reach the highest point, T. He continues to travel for another 8 seconds reaching point Q. Determine Kolapo's height above the ground when he reaches point Q.



8. (a) On Saturday, Jimmy started painting his toy helicopter between 9:00 a.m. and 10:00 a.m. When he finished between 10:00 a.m. and 11:00 a.m. on the same morning, the hour hand was exactly where the minute hand had been when he started, and the minute hand was exactly where the hour hand had been when he started. Jimmy spent t hours painting. Determine the value of t.



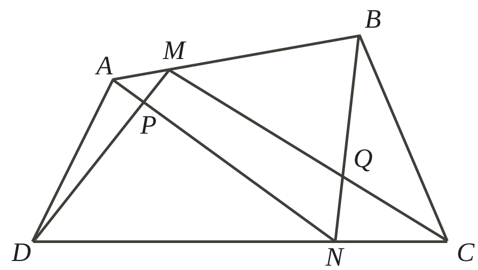
Before *After*

(b) Determine all real values of x such that

$$\log_{5x+9}(x^2 + 6x + 9) + \log_{x+3}(5x^2 + 24x + 27) = 4$$

9. (a) An auditorium has a rectangular array of chairs. There are exactly 14 boys seated in each row and exactly 10 girls seated in each column. If exactly 3 chairs are empty, prove that there are at least 567 chairs in the auditorium.

(b) In the diagram, quadrilateral $ABCD$ has points M and N on AB and DC, respectively, with $\dfrac{AM}{AB} = \dfrac{NC}{DC}$. Line segments AN and DM intersect at P, while BN and CM intersect at Q. Prove that the area of quadrilateral $PMQN$ equals the sum of the areas of $\triangle APD$ and $\triangle BQC$.



10. For each positive integer N, an *Eden sequence* from $\{1, 2, 3, \ldots, N\}$ is defined to be a sequence that satisfies the following conditions:

(i) each of its terms is an element of the set of consecutive integers $\{1, 2, 3, \ldots, N\}$,

(ii) the sequence is increasing, and

(iii) the terms in odd numbered positions are odd and the terms in even numbered positions are even.

For example, the four Eden sequences from $\{1, 2, 3\}$ are

$$1 \qquad 3 \qquad 1, 2 \qquad 1, 2, 3$$

(a) Determine the number of Eden sequences from $\{1, 2, 3, 4, 5\}$.

(b) For each positive integer N, define $e(N)$ to be the number of Eden sequences from $\{1, 2, 3, \ldots, N\}$. If $e(17) = 4180$ and $e(20) = 17710$, determine $e(18)$ and $e(19)$.

2012 Euclid Contest (English)

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

For students...

Thank you for writing the 2012 Euclid Contest!
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Euclid Contest

Tuesday, April 12, 2011

Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

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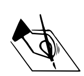
1. **FULL SOLUTION** parts are indicated like this:  .
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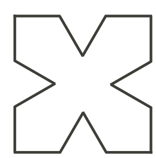
1. (a) If $(x+1)+(x+2)+(x+3)=8+9+10$, what is the value of x?

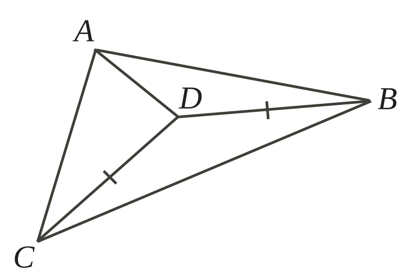
   (b) If $\sqrt{25+\sqrt{x}}=6$, what is the value of x?

   (c) The point $(a,2)$ is the point of intersection of the lines with equations $y=2x-4$ and $y=x+k$. Determine the value of k.

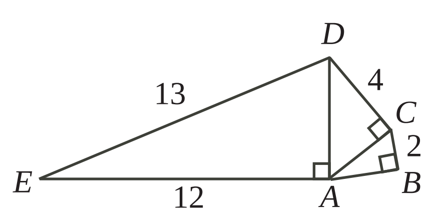
2. (a) An equilateral triangle of side length 1 is cut out of the middle of each side of a square of side length 3, as shown. What is the perimeter of the resulting figure?



   (b) In the diagram, $DC=DB$, $\angle DCB=15^\circ$, and $\angle ADB=130^\circ$. What is the measure of $\angle ADC$?



   (c) In the diagram, $\angle EAD=90^\circ$, $\angle ACD=90^\circ$, and $\angle ABC=90^\circ$. Also, $ED=13$, $EA=12$, $DC=4$, and $CB=2$. Determine the length of AB.



3. (a) If $2 \le x \le 5$ and $10 \le y \le 20$, what is the maximum value of $15 - \dfrac{y}{x}$?

(b) The functions f and g satisfy

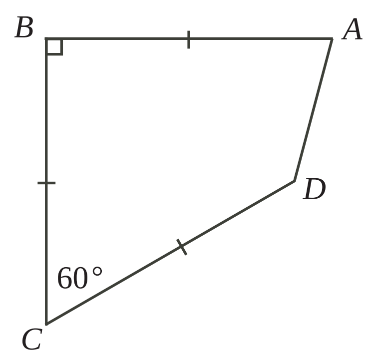
$$\begin{aligned} f(x) + g(x) &= 3x + 5 \\ f(x) - g(x) &= 5x + 7 \end{aligned}$$

for all values of x. Determine the value of $2f(2)g(2)$.

4. (a) Three different numbers are chosen at random from the set $\{1, 2, 3, 4, 5\}$.
The numbers are arranged in increasing order.
What is the probability that the resulting sequence is an arithmetic sequence?

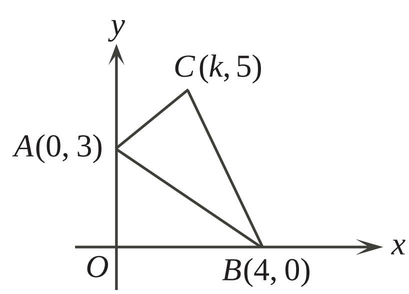
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3,5,7,9 is an arithmetic sequence with four terms.)

(b) In the diagram, $ABCD$ is a quadrilateral with $AB = BC = CD = 6$, $\angle ABC = 90°$, and $\angle BCD = 60°$. Determine the length of AD.



5. (a) What is the largest two-digit number that becomes 75% greater when its digits are reversed?

(b) A triangle has vertices $A(0, 3)$, $B(4, 0)$, $C(k, 5)$, where $0 < k < 4$. If the area of the triangle is 8, determine the value of k.



6. (a) Serge likes to paddle his raft down the Speed River from point A to point B. The speed of the current in the river is always the same. When Serge paddles, he always paddles at the same constant speed. On days when he paddles with the current, it takes him 18 minutes to get from A to B. When he does not paddle, the current carries him from A to B in 30 minutes. If there were no current, how long would it take him to paddle from A to B?

(b) Square $OPQR$ has vertices $O(0, 0)$, $P(0, 8)$, $Q(8, 8)$, and $R(8, 0)$. The parabola with equation $y = a(x - 2)(x - 6)$ intersects the sides of the square $OPQR$ at points K, L, M, and N. Determine all the values of a for which the area of the trapezoid $KLMN$ is 36.

7. (a) A 75 year old person has a 50% chance of living at least another 10 years.
A 75 year old person has a 20% chance of living at least another 15 years.
An 80 year old person has a 25% chance of living at least another 10 years.
What is the probability that an 80 year old person will live at least another 5 years?

(b) Determine all values of x for which $2^{\log_{10}(x^2)} = 3(2^{1+\log_{10} x}) + 16$.

8. The Sieve of Sundaram uses the following infinite table of positive integers:

4	7	10	13	$\cdots$
7	12	17	22	$\cdots$
10	17	24	31	$\cdots$
13	22	31	40	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The numbers in each row in the table form an arithmetic sequence. The numbers in each column in the table form an arithmetic sequence. The first four entries in each of the first four rows and columns are shown.

(a) Determine the number in the 50th row and 40th column.

(b) Determine a formula for the number in the Rth row and Cth column.

(c) Prove that if N is an entry in the table, then $2N + 1$ is composite.

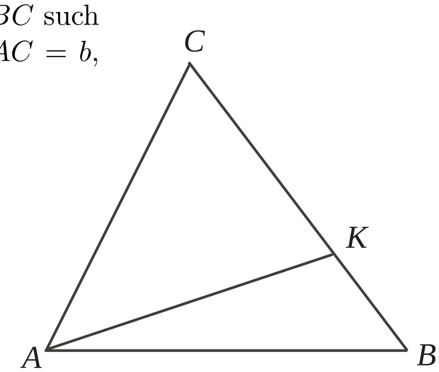
9. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$.
Suppose that $f(n) = 2n - \left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor$ and $g(n) = 2n + \left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor$ for each positive integer n.

(a) Determine the value of $g(2011)$.

(b) Determine a value of n for which $f(n) = 100$.

(c) Suppose that $A = \{f(1), f(2), f(3), \ldots\}$ and $B = \{g(1), g(2), g(3), \ldots\}$; that is, A is the range of f and B is the range of g. Prove that every positive integer m is an element of exactly one of A or B.

10. In the diagram, $2\angle BAC = 3\angle ABC$ and K lies on BC such that $\angle KAC = 2\angle KAB$. Suppose that $BC = a$, $AC = b$, $AB = c$, $AK = d$, and $BK = x$.



(a) Prove that $d = \dfrac{bc}{a}$ and $x = \dfrac{a^2 - b^2}{a}$.

(b) Prove that $(a^2 - b^2)(a^2 - b^2 + ac) = b^2c^2$.

(c) Determine a triangle with positive integer side lengths a, b, c and positive area that satisfies the condition in part (b).

2011
Euclid
Contest
(English)

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Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest

Wednesday, April 7, 2010

Time: $2\frac{1}{2}$ hours

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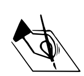
1. **FULL SOLUTION** parts are indicated like this:  .
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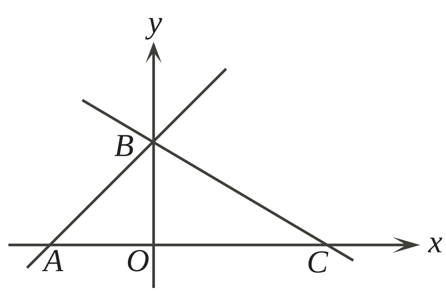
1. (a) If $3^x = 27$, what is the value of 3^{x+2}?

   (b) If $2^5 3^{13} 5^9 x = 2^7 3^{14} 5^9$, what is the value of x?

   (c) Triangle ABC is enclosed by the lines $y = x + 2$, $y = -\frac{1}{2}x + 2$ and the x-axis. Determine the area of $\triangle ABC$.

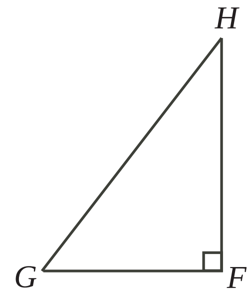


2. (a) Maria has a red package, a green package, and a blue package.
   The sum of the masses of the three packages is 60 kg.
   The sum of the masses of the red and green packages is 25 kg.
   The sum of the masses of the green and blue packages is 50 kg.
   What is the mass of the green package, in kg?

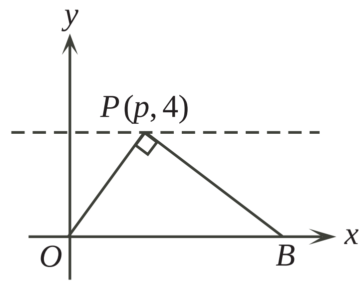
   (b) A *palindrome* is a positive integer that is the same when read forwards or backwards. For example, 151 is a palindrome. What is the largest palindrome less than 200 that is the sum of three consecutive integers?

   (c) If $(x + 1)(x - 1) = 8$, determine the numerical value of $(x^2 + x)(x^2 - x)$.

3. (a) Bea the bee sets out from her hive, H, and flies south for 1 hour to a field, F. She spends 30 minutes in the field, and then flies 45 minutes west to a garden, G. After spending 1 hour in the garden, she flies back to her hive along a straight line route. Bea always flies at the same constant speed. What is the total length of time, in minutes, that she is away from her hive?



(b) In the diagram, points $P(p,4)$, $B(10,0)$, and $O(0,0)$ are shown. If $\triangle OPB$ is right-angled at P, determine all possible values of p.



4. (a) Thurka bought some stuffed goats and some toy helicopters. She paid a total of \$201. She did not buy partial goats or partial helicopters. Each stuffed goat cost \$19 and each toy helicopter cost \$17. How many of each did she buy?

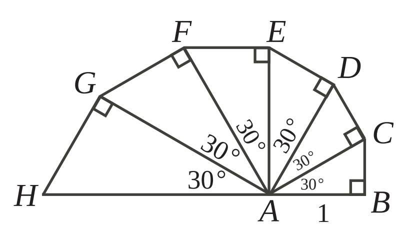
(b) Determine all real values of x for which $(x+8)^4 = (2x+16)^2$.

5. (a) If $f(x) = 2x+1$ and $g(f(x)) = 4x^2+1$, determine an expression for $g(x)$.

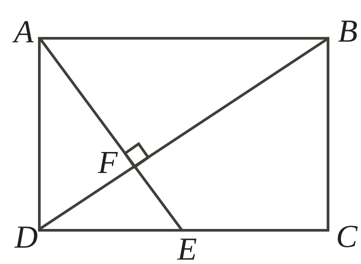
(b) A geometric sequence has 20 terms.
The sum of its first two terms is 40.
The sum of its first three terms is 76.
The sum of its first four terms is 130.
Determine how many of the terms in the sequence are integers.

(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

6. (a) A snail's shell is formed from six triangular sections, as shown. Each triangle has interior angles of 30°, 60° and 90°. If AB has a length of 1 cm, what is the length of AH, in cm?



(b) In rectangle $ABCD$, point E is on side DC. Line segments AE and BD are perpendicular and intersect at F. If $AF = 4$ and $DF = 2$, determine the area of quadrilateral $BCEF$.

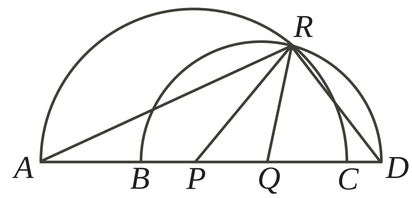


7. (a) Determine all real values of x for which $3^{(x-1)} 9^{\frac{3}{2x^2}} = 27$.

(b) Determine all points (x, y) where the two curves $y = \log_{10}(x^4)$ and $y = (\log_{10} x)^3$ intersect.

8. (a) Oi-Lam tosses three fair coins and removes all of the coins that come up heads. George then tosses the coins that remain, if any. Determine the probability that George tosses exactly one head.

(b) In the diagram, points B, P, Q, and C lie on line segment AD. The semi-circle with diameter AC has centre P and the semi-circle with diameter BD has centre Q. The two semi-circles intersect at R. If $\angle PRQ = 40^\circ$, determine the measure of $\angle ARD$.



9. (a) (i) If θ is an angle whose measure is not an integer multiple of 90°, prove that

$$\cot\theta - \cot 2\theta = \frac{1}{\sin 2\theta}$$

(ii) Ross starts with an angle of measure 8° and doubles it 10 times until he obtains 8192°. He then adds up the reciprocals of the sines of these 11 angles. That is, he calculates

$$S = \frac{1}{\sin 8^\circ} + \frac{1}{\sin 16^\circ} + \frac{1}{\sin 32^\circ} + \cdots + \frac{1}{\sin 4096^\circ} + \frac{1}{\sin 8192^\circ}$$

Determine, without using a calculator, the measure of the acute angle α so that $S = \dfrac{1}{\sin\alpha}$.

(b) In $\triangle ABC$, $BC = a$, $AC = b$, $AB = c$, and $a < \frac{1}{2}(b + c)$.
Prove that $\angle BAC < \frac{1}{2}(\angle ABC + \angle ACB)$.

10. For each positive integer n, let $T(n)$ be the number of triangles with integer side lengths, positive area, and perimeter n. For example, $T(6) = 1$ since the only such triangle with a perimeter of 6 has side lengths 2, 2 and 2.

(a) Determine the values of $T(10)$, $T(11)$ and $T(12)$.

(b) If m is a positive integer with $m \geq 3$, prove that $T(2m) = T(2m - 3)$.

(c) Determine the smallest positive integer n such that $T(n) > 2010$.

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Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest

Tuesday, April 7, 2009

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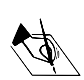
1. **FULL SOLUTION** parts are indicated like this:  .
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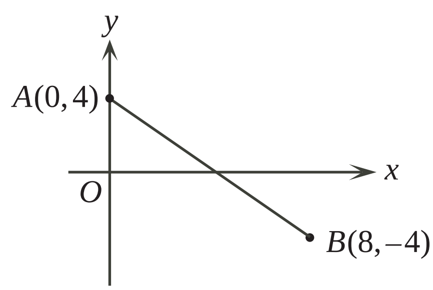
1. (a) A line has equation $6x + 3y - 21 = 0$. What is the slope of the line?

   (b) A line with a slope of 3 passes through the points $(1, 0)$ and $(5, c)$. What is the value of c?

   (c) The point (k, k) lies on the line segment AB shown in the diagram. Determine the value of k.

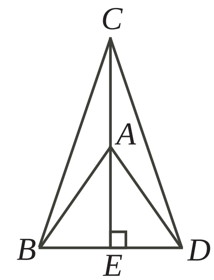


2. (a) What is the sum of the two numbers that satisfy the equation $x^2 - 6x - 7 = 0$?

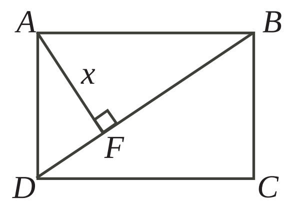
   (b) What is the product of the two numbers that satisfy the equation $5x^2 - 20 = 0$?

   (c) Determine the average of the numbers that satisfy the equation $x^3 - 6x^2 + 5x = 0$.

3. (a) In the diagram, $AB = AC = AD = BD$ and CAE is a straight line segment that is perpendicular to BD. What is the measure of $\angle CDB$?



(b) In rectangle $ABCD$, F is on diagonal BD so that AF is perpendicular to BD. Also, $BC = 30$, $CD = 40$ and $AF = x$. Determine the value of x.

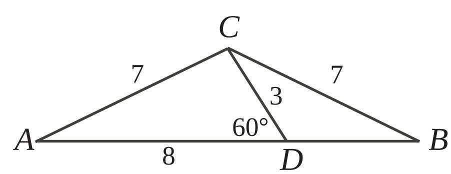


4. (a) In an arithmetic sequence, the first term is 1 and the last term is 19. The sum of all the terms in the sequence is 70. How many terms does the sequence have?

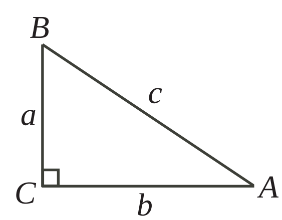
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9 is an arithmetic sequence with four terms.)

(b) Suppose that $a(x + b(x + 3)) = 2(x + 6)$ for *all* values of x. Determine a and b.

5. (a) In the diagram, $\triangle ABC$ is isosceles with $AC = BC = 7$. Point D is on AB with $\angle CDA = 60°$, $AD = 8$, and $CD = 3$. Determine the length of BD.

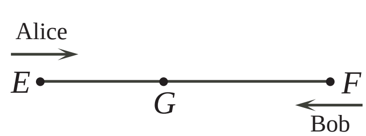


(b) In the diagram, $\triangle ABC$ is right-angled at C. Also, $2 \sin B = 3 \tan A$. Determine the measure of angle A.



6. (a) An integer n, with $100 \leq n \leq 999$, is chosen at random. What is the probability that the sum of the digits of n is 24?

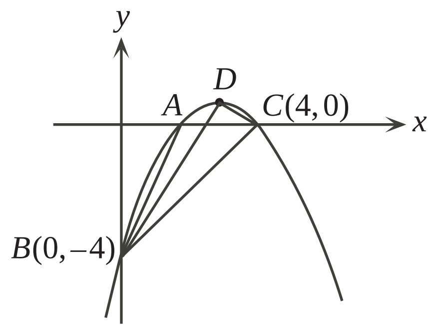
(b) Alice drove from town E to town F at a constant speed of 60 km/h. Bob drove from F to E along the same road also at a constant speed. They started their journeys at the same time and passed each other at point G.



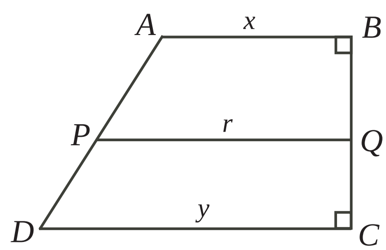
Alice drove from G to F in 45 minutes. Bob drove from G to E in 20 minutes. Determine Bob's constant speed.

7\. (a) The parabola $y = x^2 - 2x + 4$ is translated p units to the right and q units down. The x-intercepts of the resulting parabola are 3 and 5. What are the values of p and q?

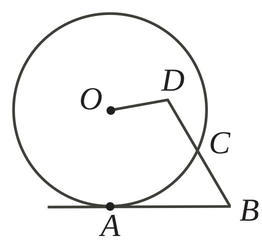
(b) In the diagram, D is the vertex of a parabola. The parabola cuts the x-axis at A and at $C(4,0)$. The parabola cuts the y-axis at $B(0,-4)$. The area of $\triangle ABC$ is 4. Determine the area of $\triangle DBC$.



8\. (a) $ABCD$ is a trapezoid with parallel sides AB and DC. Also, BC is perpendicular to AB and to DC. The line PQ is parallel to AB and divides the trapezoid into two regions of equal area. If $AB = x$, $DC = y$, and $PQ = r$, prove that $x^2 + y^2 = 2r^2$.



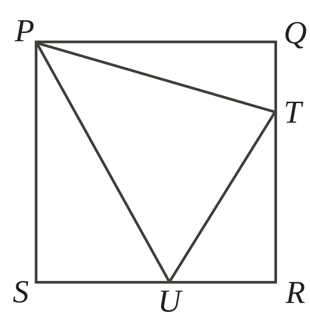
(b) In the diagram, AB is tangent to the circle with centre O and radius r. The length of AB is p. Point C is on the circle and D is inside the circle so that BCD is a straight line, as shown. If $BC = CD = DO = q$, prove that $q^2 + r^2 = p^2$.



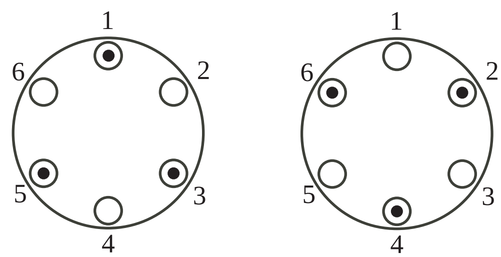
9\. (a) If $\log_2 x$, $(1 + \log_4 x)$, and $\log_8 4x$ are consecutive terms of a geometric sequence, determine the possible values of x.

(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

(b) In the diagram, $PQRS$ is a square with sides of length 4. Points T and U are on sides QR and RS respectively such that $\angle UPT = 45^\circ$. Determine the maximum possible perimeter of $\triangle RUT$.



10. Suppose there are n plates equally spaced around a circular table. Ross wishes to place an identical gift on each of k plates, so that no two neighbouring plates have gifts. Let $f(n, k)$ represent the number of ways in which he can place the gifts. For example $f(6, 3) = 2$, as shown below.



(a) Determine the value of $f(7, 3)$.

(b) Prove that $f(n, k) = f(n-1, k) + f(n-2, k-1)$ for all integers $n \geq 3$ and $k \geq 2$.

(c) Determine the smallest possible value of $n + k$ among all possible ordered pairs of integers (n, k) for which $f(n, k)$ is a positive multiple of 2009, where $n \geq 3$ and $k \geq 2$.

Canadian Mathematics Competition

For students...

Thank you for writing the 2009 Euclid Contest!
In 2008, more than 14 000 students from around the world registered to write the Euclid Contest.

If you are graduating from secondary school, good luck in your future endeavours!
If you will be returning to secondary school next year, encourage your teacher to register you for the 2009 Sun Life Financial Canadian Open Mathematics Challenge, which will be written in late November.
Visit our website

www.cemc.uwaterloo.ca

to find

- **More information about the Sun Life Financial Canadian Open Mathematics Challenge**
- **Free copies of past contests**
- **Workshops to help you prepare for future contests**
- **Information about our publications for mathematics enrichment and contest preparation**

For teachers...

Visit our website

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to

- **Obtain information about our 2009/2010 contests**
- **Learn about workshops and resources we offer for teachers**
- **Find your school results**

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest

Tuesday, April 15, 2008

Time: $2\frac{1}{2}$ hours

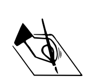
Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 3 marks each. **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

Instructions for SHORT ANSWER parts:

1. **SHORT ANSWER** parts are indicated like this:  .
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

Instructions for FULL SOLUTION parts:

1. **FULL SOLUTION** parts are indicated like this:  .
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

NOTE: At the completion of the Contest, insert the information sheet inside the answer booklet.

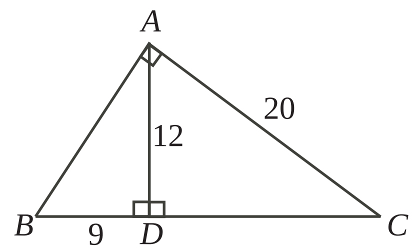
The names of some top-scoring students will be published in the Euclid Results on our Web site, http://www.cemc.uwaterloo.ca.

NOTES:	1.	Please read the instructions on the front cover of this booklet.
	2.	Write all answers in the answer booklet provided.
	3.	For questions marked “ ”, full marks will be given for a correct answer placed in the appropriate box in the answer booklet. **If an incorrect answer is given, marks may be given for work shown.** Students are strongly encouraged to show their work.
	4.	All calculations and answers should be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., rather than as $12.566\ldots$ or $4.646\ldots$, except where otherwise indicated.

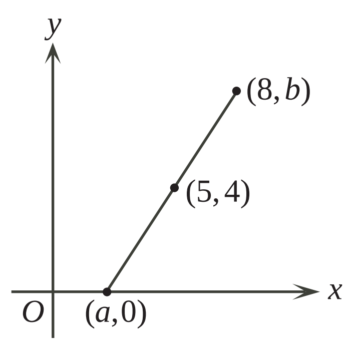
A Note about Writing Solutions

For each problem marked “ ”, a full solution is required. The solutions that you provide in the answer booklet should be well organized and contain a balance of mathematical statements and words of explanation. Working out some of the details in rough on a separate piece of paper before writing your finished solution is a good idea. Your final solution should be written so that the marker can understand your approach to the problem and all of the mathematical steps of your solution.

1. (a) In the diagram, what is the perimeter of $\triangle ABC$?



(b) In the diagram, the line segment with endpoints $(a,0)$ and $(8,b)$ has midpoint $(5,4)$. What are the values of a and b?



(c) The lines $ax+y=30$ and $x+ay=k$ intersect at the point $P(6,12)$. Determine the value of k.

2. Each part of this problem refers to the parabola $y=(x-2)(x-8)+7$.

(a) The points $(2,7)$ and $(c,7)$, where $c\neq 2$, lie on the parabola. What is the value of c?

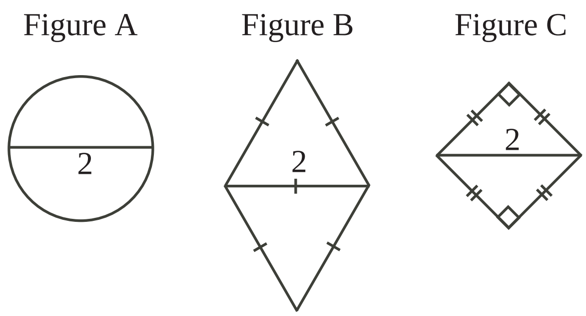
(b) What are the coordinates of the vertex of the parabola?

(c) A line that passes through the point $A(5,0)$ intersects the parabola at $B(4,-1)$. Determine the other point at which this line intersects the parabola.

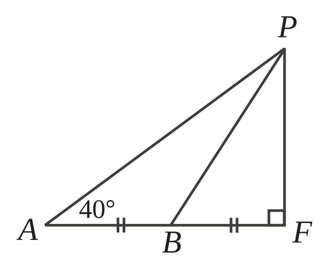
3. (a) A 3×3 square frame is placed on a grid of numbers, as shown. In the example, the sum of the numbers inside the square frame is 108, and the middle number is 12. When the square frame is moved to a new position, the sum of its numbers becomes 279. In the frame's new position, what is the middle number?

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

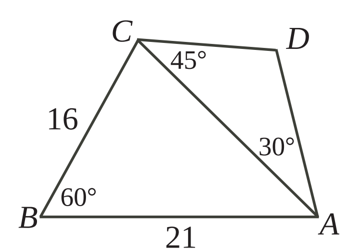
(b) Of the three figures shown, which has the smallest area and which has the largest area? Explain how you determined your answer. (In Figure A, the circle has a diameter of length 2.)



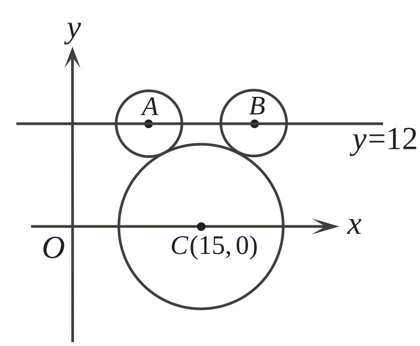
4. (a) A flagpole FP is 20 metres tall. From point A on the flat ground, the angle of elevation to the top of the flagpole is $40°$. If B is halfway from A to F, what is the measure of $\angle FBP$, to the nearest degree?



(b) In the diagram, $AB = 21$ and $BC = 16$. Also, $\angle ABC = 60°$, $\angle CAD = 30°$, and $\angle ACD = 45°$. Determine the length of CD, to the nearest tenth.



5. (a) In the diagram, the large circle has radius 9 and centre $C(15, 0)$. The small circles have radius 4 and centres A and B on the horizontal line $y = 12$. Each of the two small circles is tangent to the large circle. It takes a bug 5 seconds to walk at a constant speed from A to B along the line $y = 12$. How far does the bug walk in 1 second?



(b) Determine all values of k, with $k \neq 0$, for which the parabola

$$y = kx^2 + (5k+3)x + (6k+5)$$

has its vertex on the x-axis.

6. (a) The function $f(x)$ satisfies the equation $f(x) = f(x-1) + f(x+1)$ for all values of x. If $f(1) = 1$ and $f(2) = 3$, what is the value of $f(2008)$?

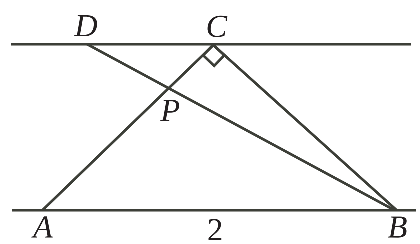
(b) The numbers a, b, c, in that order, form a three term arithmetic sequence (see below) and $a + b + c = 60$.
The numbers $a - 2$, b, $c + 3$, in that order, form a three term geometric sequence.
Determine all possible values of a, b and c.

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7 is an arithmetic sequence with three terms.
A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

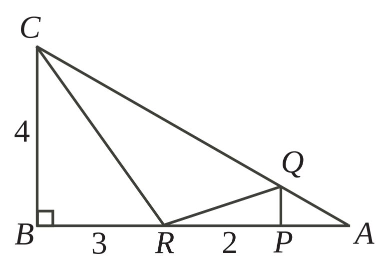
7. (a) The average of three consecutive multiples of 3 is a.
The average of four consecutive multiples of 4 is $a + 27$.
The average of the smallest and largest of these seven integers is 42.
Determine the value of a.

(b) Billy and Crystal each have a bag of 9 balls. The balls in each bag are numbered from 1 to 9. Billy and Crystal each remove one ball from their own bag. Let b be the sum of the numbers on the balls remaining in Billy's bag. Let c be the sum of the numbers on the balls remaining in Crystal's bag. Determine the probability that b and c differ by a multiple of 4.

8. (a) Points A, B, C, and D are arranged, as shown, with AB parallel to DC and P the point of intersection of AC and BD. Also, $\angle ACB = 90°$, $AC = CB$, $AB = BD = 2$. Determine the measure of $\angle DBC$.



(b) In the diagram, ABC is a right-angled triangle with P and R on AB. Also, Q is on AC, and PQ is parallel to BC. If $RP = 2$, $BR = 3$, $BC = 4$, and the area of $\triangle QRC$ is 5, determine the length of AP.



9. (a) The equation $2^{x+2}5^{6-x} = 10^{x^2}$ has two real solutions. Determine these two solutions.

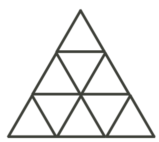
(b) Determine all real solutions to the system of equations

$$\begin{array}{rcrcl} x & + & \log_{10} x & = & y-1 \\ y & + & \log_{10}(y-1) & = & z-1 \\ z & + & \log_{10}(z-2) & = & x+2 \end{array}$$

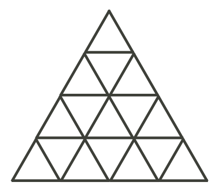
and prove that there are no more solutions.

10. Suppose that n is a positive integer. Consider an upward-pointing equilateral triangle of side length n, cut up into unit triangles, as shown.

$n = 3$

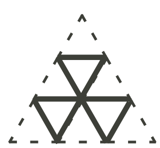


$n = 4$



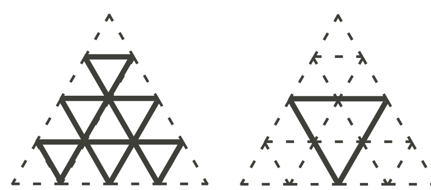
For each n, let $f(n)$ represent the total number of downward-pointing equilateral triangles of *all* sizes. For example, $f(3) = 3$ and $f(4) = 6 + 1 = 7$, as illustrated below.

$n = 3$



$f(3) = 3$

$n = 4$



$f(4) = 6 + 1 = 7$

(a) Determine the values of $f(5)$ and $f(6)$.

(b) Prove that $f(2k) = f(2k-1) + k^2$ for each positive integer $k \geq 1$.

(c) Determine, with justification, all positive integers n for which $f(n)$ is divisible by n.

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- Obtain information about our 2008/2009 contests
- Learn about workshops and resources we offer for teachers
- Find your school results

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest

Tuesday, April 17, 2007

Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

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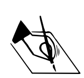
1. **FULL SOLUTION** parts are indicated like this:  .
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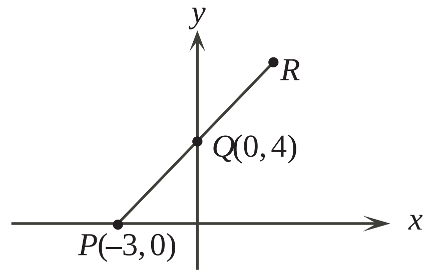
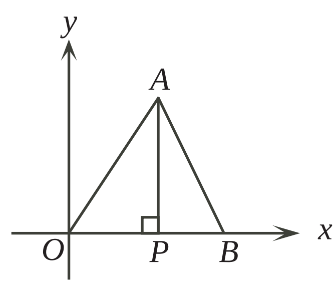
1. (a) If the point $(a-1, a+1)$ lies on the line $y = 2x - 3$, what is the value of a?

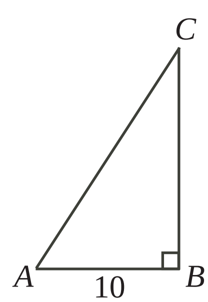
   (b) In the diagram, a line is drawn through points P, Q and R. If $PQ = QR$, what are the coordinates of R?



   (c) In the diagram, $OA = 15$, $OP = 9$ and $PB = 4$. Determine the equation of the line through A and B. Explain how you got your answer.

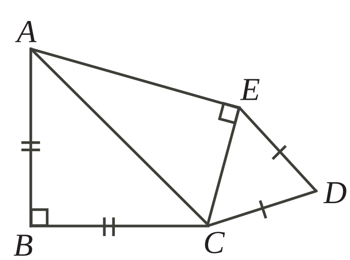


2. (a) In the diagram, $\triangle ABC$ is right-angled at B and $AB = 10$. If $\cos(\angle BAC) = \frac{5}{13}$, what is the value of $\tan(\angle ACB)$?



   (b) Suppose $0^\circ < x < 90^\circ$ and $2\sin^2 x + \cos^2 x = \frac{25}{16}$. What is the value of $\sin x$?

   (c) In the diagram, $AB = BC = 2\sqrt{2}$, $CD = DE$, $\angle CDE = 60^\circ$, and $\angle EAB = 75^\circ$. Determine the perimeter of figure $ABCDE$. Explain how you got your answer.

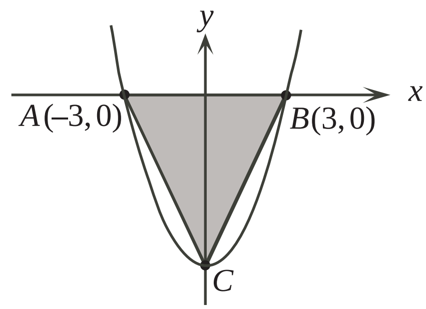


3. (a) The first term of a sequence is 2007. Each term, starting with the second, is the sum of the cubes of the digits of the previous term. What is the 2007th term?

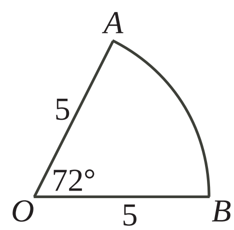
(b) Sequence A has nth term $n^2 - 10n + 70$.
(The first three terms of sequence A are 61, 54, 49.)
Sequence B is an arithmetic sequence with first term 5 and common difference 10.
(The first three terms of sequence B are 5, 15, 25.)
Determine all n for which the nth term of sequence A is equal to the nth term of sequence B. Explain how you got your answer.

4. (a) Determine all values of x for which $2 + \sqrt{x-2} = x - 2$.

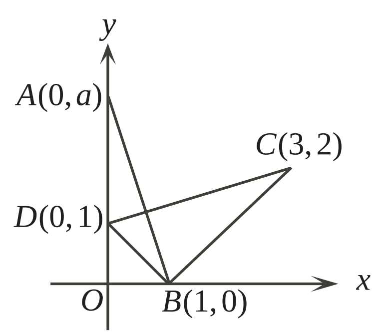
(b) In the diagram, the parabola intersects the x-axis at $A(-3,0)$ and $B(3,0)$ and has its vertex at C below the x-axis. The area of $\triangle ABC$ is 54. Determine the equation of the parabola. Explain how you got your answer.



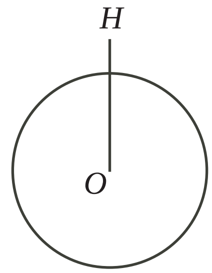
5. (a) In the diagram, a sector of a circle with centre O and radius 5 is shown. What is the perimeter of the sector?



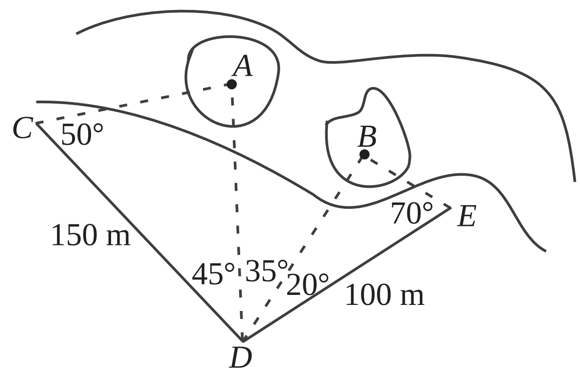
(b) In the diagram, $A(0,a)$ lies on the y-axis above D. If the triangles AOB and BCD have the same area, determine the value of a. Explain how you got your answer.



6. (a) The Little Prince lives on a spherical planet which has a radius of 24 km and centre O. He hovers in a helicopter (H) at a height of 2 km above the surface of the planet. From his position in the helicopter, what is the distance, in kilometres, to the furthest point on the surface of the planet that he can see?

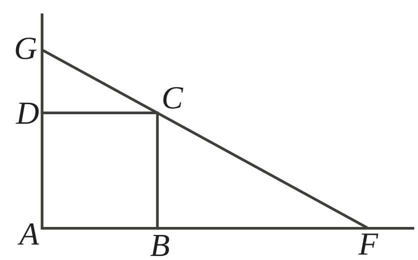


(b) In the diagram, points A and B are located on islands in a river full of rabid aquatic goats. Determine the distance from A to B, to the nearest metre. (Luckily, someone has measured the angles shown in the diagram as well as the distances CD and DE.)

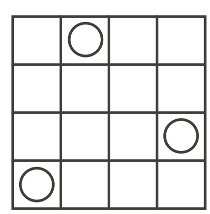


7. (a) Determine all values of x for which $(\sqrt{x})^{\log_{10} x} = 100$.

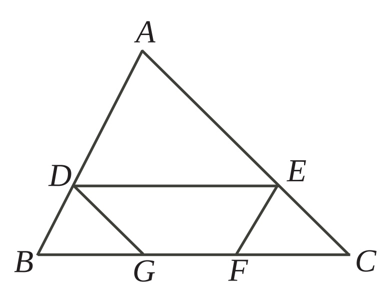
(b) In the diagram, line segment FCG passes through vertex C of square $ABCD$, with F lying on AB extended and G lying on AD extended. Prove that $\dfrac{1}{AB} = \dfrac{1}{AF} + \dfrac{1}{AG}$.



8. (a) In the 4×4 grid shown, three coins are randomly placed in different squares. Determine the probability that no two coins lie in the same row or column.



(b) In the diagram, the area of $\triangle ABC$ is 1. Trapezoid $DEFG$ is constructed so that G is to the left of F, DE is parallel to BC, EF is parallel to AB and DG is parallel to AC. Determine the maximum possible area of trapezoid $DEFG$.

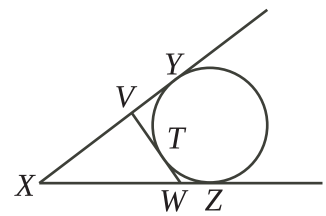


9. The parabola $y = f(x) = x^2 + bx + c$ has vertex P and the parabola $y = g(x) = -x^2 + dx + e$ has vertex Q, where P and Q are distinct points. The two parabolas also intersect at P and Q.

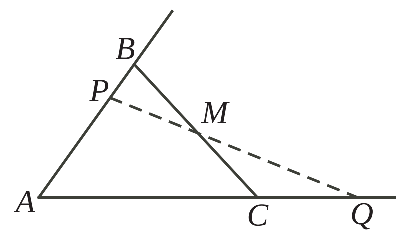
(a) Prove that $2(e - c) = bd$.

(b) Prove that the line through P and Q has slope $\frac{1}{2}(b + d)$ and y-intercept $\frac{1}{2}(c + e)$.

10. (a) In the diagram, the circle is tangent to XY at Y and to XZ at Z. Point T is chosen on the minor arc YZ and a tangent to the circle is drawn at T, cutting XY at V and XZ at W. Prove that the perimeter of $\triangle VXW$ is independent of the position of T.



(b) In the diagram, $AB = 10$, $BC = 14$, $AC = 16$, and M is the midpoint of BC. Various lines can be drawn through M, cutting AB (possibly extended) at P and AC (possibly extended) at Q. Determine, with proof, the minimum possible perimeter of $\triangle APQ$.



Canadian Mathematics Competition

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Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest

Wednesday, April 19, 2006

Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

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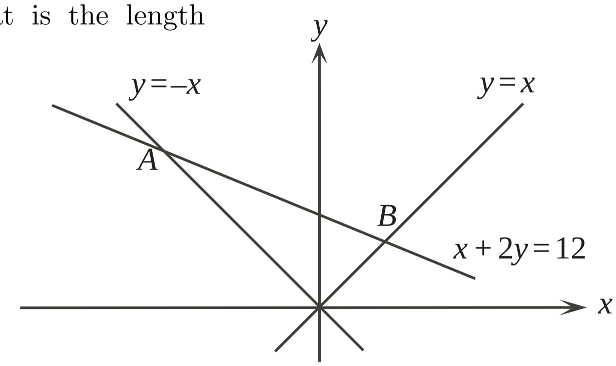
4. All calculations and answers should be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., except where otherwise indicated.

1\. (a) What is the sum of the x-intercept and the y-intercept of the line $3x-3y=24$?

(b) If the lines $px=12$ and $2x+qy=10$ intersect at $(1,1)$, what is the value of $p+q$?

(c) In the diagram, the line $x+2y=12$ intersects the lines $y=-x$ and $y=x$ at points A and B, respectively. What is the length of AB?



2\. (a) The average of the digits of the integer 46 is 5. Including 46, how many two-digit positive integers have the average of their digits equal to 5?

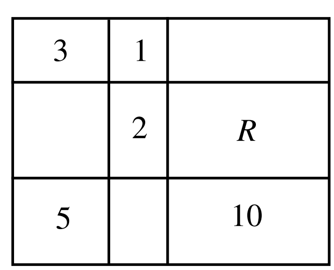
(b) When a decimal point is placed between the digits of the two-digit integer n, the resulting number is equal to the average of the digits of n. What is the value of n?

(c) The average of three positive integers is 28. When two additional integers, s and t, are included, the average of all five integers is 34. What is the average of s and t?

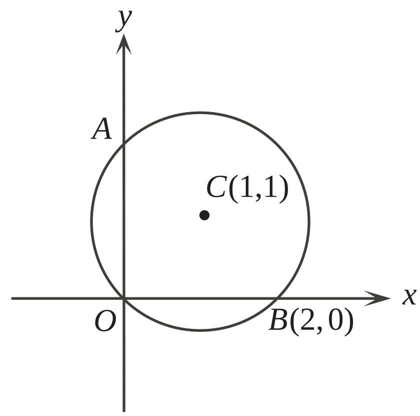
3\. (a) Determine the coordinates of the vertex of the parabola $y=(x-20)(x-22)$.

(b) Point A is the vertex of the parabola $y=x^2+2$, point B is the vertex of the parabola $y=x^2-6x+7$, and O is the origin. Determine the area of $\triangle OAB$.

4\. (a) In the diagram, the rectangle is divided into nine smaller rectangles. The areas of five of these rectangles are given. Determine the area of the rectangle labelled R.



(b) In the diagram, the circle with centre $C(1,1)$ passes through the point $O(0,0)$, intersects the y-axis at A, and intersects the x-axis at $B(2,0)$. Determine, with justification, the coordinates of A and the area of the part of the circle that lies in the first quadrant.



5. (a) If a is chosen randomly from the set $\{1,2,3,4,5\}$ and b is chosen randomly from the set $\{6,7,8\}$, what is the probability that a^b is an even number?

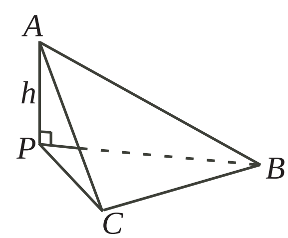
(b) A bag contains some blue and some green hats. On each turn, Julia removes one hat without looking, with each hat in the bag being equally likely to be chosen. If it is green, she adds a blue hat into the bag from her supply of extra hats, and if it is blue, she adds a green hat to the bag. The bag initially contains 4 blue hats and 2 green hats. What is the probability that the bag again contains 4 blue hats and 2 green hats after two turns?

6. (a) Suppose that, for some angles x and y,

$$\begin{aligned} \sin^2 x + \cos^2 y &= \tfrac{3}{2}a \\ \cos^2 x + \sin^2 y &= \tfrac{1}{2}a^2 \end{aligned}$$

Determine the possible value(s) of a.

(b) Survivors on a desert island find a piece of plywood (ABC) in the shape of an equilateral triangle with sides of length 2 m. To shelter their goat from the sun, they place edge BC on the ground, lift corner A, and put in a vertical post PA which is h m long above ground. When the sun is directly overhead, the shaded region ($\triangle PBC$) on the ground directly underneath the plywood is an isosceles triangle with largest angle ($\angle BPC$) equal to 120°. Determine the value of h, to the nearest centimetre.

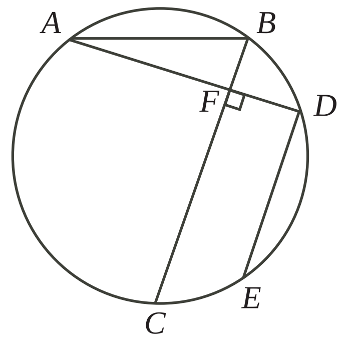


7. (a) The sequence 2, 5, 10, 50, 500, $\ldots$ is formed so that each term after the second is the product of the two previous terms. The 15th term ends with exactly k zeroes. What is the value of k?

(b) Suppose that a, b, c are three consecutive terms in an arithmetic sequence. Prove that $a^2 - bc$, $b^2 - ac$, and $c^2 - ab$ are also three consecutive terms in an arithmetic sequence.
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7 is an arithmetic sequence with three terms.)

8\. (a) If $\log_2 x - 2\log_2 y = 2$, determine y as a function of x, and sketch a graph of this function on the axes in the answer booklet.

(b) In the diagram, AB and BC are chords of the circle with $AB < BC$. If D is the point on the circle such that AD is perpendicular to BC and E is the point on the circle such that DE is parallel to BC, carefully prove, explaining all steps, that $\angle EAC + \angle ABC = 90^\circ$.



9\. Define $f(x) = \sin^6 x + \cos^6 x + k\left(\sin^4 x + \cos^4 x\right)$ for some real number k.

(a) Determine all real numbers k for which $f(x)$ is constant for all values of x.

(b) If $k = -0.7$, determine all solutions to the equation $f(x) = 0$.

(c) Determine all real numbers k for which there exists a real number c such that $f(c) = 0$.

10\. Points A_1, A_2, …, A_N are equally spaced around the circumference of a circle and $N \geq 3$. Three of these points are selected at random and a triangle is formed using these points as its vertices.

(a) If $N = 7$, what is the probability that the triangle is acute? (A triangle is acute if each of its three interior angles is less than 90°.)

(b) If $N = 2k$ for some positive integer $k \geq 2$, determine the probability that the triangle is acute.

(c) If $N = 2k$ for some positive integer $k \geq 2$, determine all possible values of k for which the probability that the triangle is acute can be written in the form $\dfrac{a}{2007}$ for some positive integer a.

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Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest

for
The CENTRE for EDUCATION in MATHEMATICS and COMPUTING Awards

Tuesday, April 19, 2005

C.M.C. Sponsors:

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Canadian Institute of Actuaries

Great West Life and London Life

Sybase

iAnywhere Solutions

Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphics displays.

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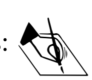
1. **FULL SOLUTION** parts are indicated like this:  .
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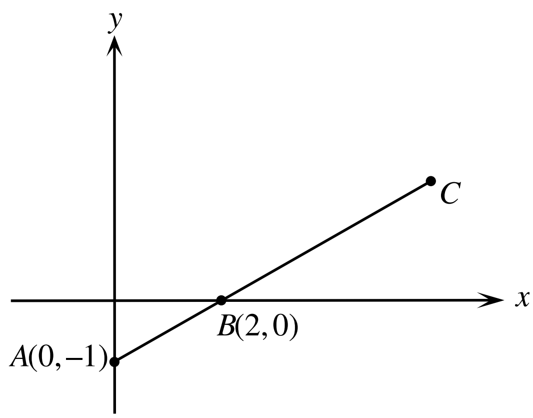
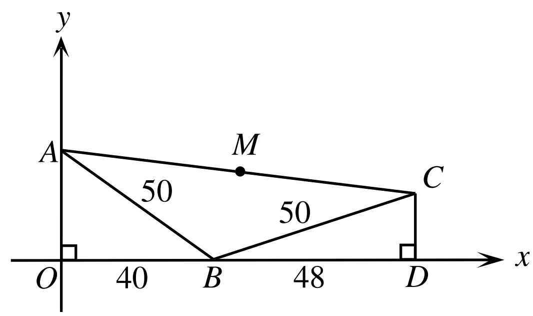
1. (a) If the point (a,a) lies on the line $3x-y=10$, what is the value of a?

(b) In the diagram, points A, B and C lie on a line such that $BC=2AB$. What are the coordinates of C?



(c) In the diagram, triangles AOB and CDB are right-angled and M is the midpoint of AC. What are the coordinates of M?



2. (a) If $y=2x+3$ and $4y=5x+6$, what is the value of x?

(b) If a, b and c are numbers such that

$$\begin{aligned} -3b+7c &= -10 \\ b-2c &= 3 \\ a+2b-5c &= 13 \end{aligned}$$

what is the value of a?

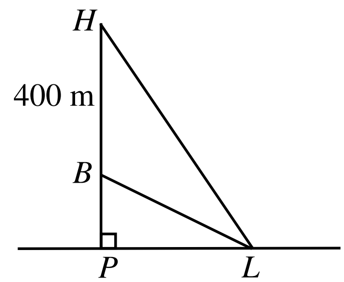
(c) John and Mary wrote the Euclid Contest. Two times John's score was 60 more than Mary's score. Two times Mary's score was 90 more than John's score. Determine the average of their two scores.

3. (a) If $2^x = 2(16^{12}) + 2(8^{16})$, what is the value of x?

(b) If $f(x)=2x-1$, determine all real values of x such that $\left(f(x)\right)^2 - 3f(x) + 2 = 0$.

4. (a) Six tickets numbered 1 through 6 are placed in a box. Two tickets are randomly selected and removed together. What is the probability that the smaller of the two numbers on the tickets selected is less than or equal to 4?

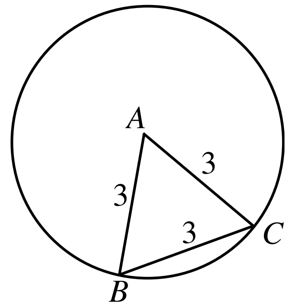
(b) A helicopter hovers at point H, directly above point P on level ground. Lloyd sits on the ground at a point L where $\angle HLP = 60^\circ$. A ball is droppped from the helicopter. When the ball is at point B, 400 m directly below the helicopter, $\angle BLP = 30^\circ$. What is the distance between L and P?



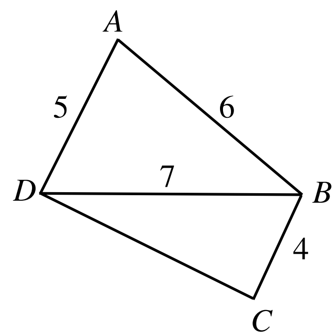
5. (a) A goat starts at the origin $(0, 0)$ and then makes several moves. On move 1, it travels 1 unit up to $(0, 1)$. On move 2, it travels 2 units right to $(2, 1)$. On move 3, it travels 3 units down to $(2, -2)$. On move 4, it travels 4 units to $(-2, -2)$. It continues in this fashion, so that on move n, it turns 90° in a clockwise direction from its previous heading and travels n units in this new direction. After n moves, the goat has travelled a total of 55 units. Determine the coordinates of its position at this time.

(b) Determine all possible values of r such that the three term geometric sequence 4, $4r$, $4r^2$ is also an arithmetic sequence.
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9, 11 is an arithmetic sequence.)

6. (a) Equilateral triangle ABC has side length 3, with vertices B and C on a circle of radius 3, as shown. The triangle is then rotated clockwise inside the circle: first about C until A reaches the circle, and then about A until B reaches the circle, and so on. Eventually the triangle returns to its original position and stops. What is the total distance travelled by the point B?



(b) In the diagram, $ABCD$ is a quadrilateral in which $\angle A + \angle C = 180°$. What is the length of CD?

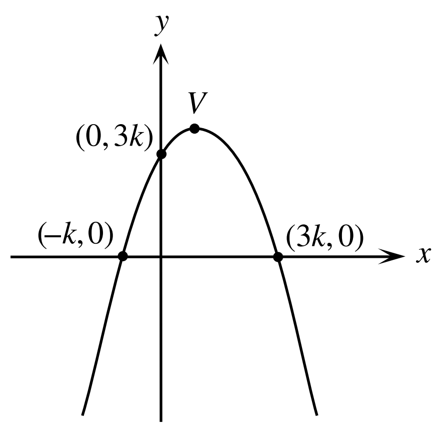


7. (a) If $f(x) = \sin^2 x - 2\sin x + 2$, what are the minimum and maximum values of $f(x)$?

(b) In the diagram, the parabola

$$y = -\frac{1}{4}(x-r)(x-s)$$

intersects the axes at three points. The vertex of this parabola is the point V. Determine the value of k and the coordinates of V.

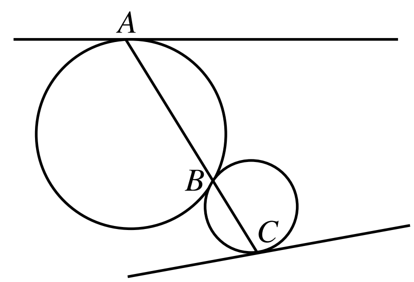


8. (a) A function is defined by

$$f(x)=\begin{cases} 4 & \text{if } x<-4 \\ -x & \text{if } -4\le x\le 5 \\ -5 & \text{if } x>5 \end{cases}$$

On the grid in the answer booklet, sketch the graph $g(x)=\sqrt{25-[f(x)]^2}$. State the shape of each portion of the graph.

(b) In the diagram, two circles are tangent to each other at point B. A straight line is drawn through B cutting the two circles at A and C, as shown. Tangent lines are drawn to the circles at A and C. Prove that these two tangent lines are parallel.



9. The circle $(x-p)^2+y^2=r^2$ has centre C and the circle $x^2+(y-p)^2=r^2$ has centre D. The circles intersect at two *distinct* points A and B, with x-coordinates a and b, respectively.
(a) Prove that $a+b=p$ and $a^2+b^2=r^2$.
(b) If r is fixed and p is then found to maximize the area of quadrilateral $CADB$, prove that either A or B is the origin.
(c) If p and r are integers, determine the minimum possible distance between A and B. Find positive integers p and r, each larger than 1, that give this distance.

10. A school has a row of n open lockers, numbered 1 through n. After arriving at school one day, Josephine starts at the beginning of the row and closes every second locker until reaching the end of the row, as shown in the example below. Then on her way back, she closes every second locker that is still open. She continues in this manner along the row, until only one locker remains open. Define $f(n)$ to be the number of the last open locker. For example, if there are 15 lockers, then $f(15)=11$ as shown below:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
$\longrightarrow$	1	$\cancel{2}$	3	$\cancel{4}$	5	$\cancel{6}$	7	$\cancel{8}$	9	$\cancel{10}$	11	$\cancel{12}$	13	$\cancel{14}$	15	
	$\cancel{1}$		3		$\cancel{5}$		7		$\cancel{9}$		11		$\cancel{13}$		15	$\longleftarrow$
$\longrightarrow$			3				$\cancel{7}$				11				$\cancel{15}$	
			$\cancel{3}$								11					$\longleftarrow$

(a) Determine $f(50)$.
(b) Prove that there is no positive integer n such that $f(n)=2005$.
(c) Prove that there are infinitely many positive integers n such that $f(n)=f(2005)$.

2005
Euclid
Contest
(English)

Canadian Mathematics Competition

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Mathematics
Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

Wednesday, April 14, 2004

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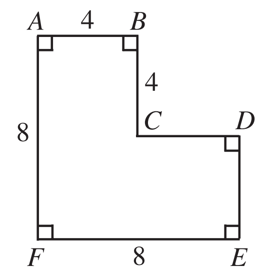
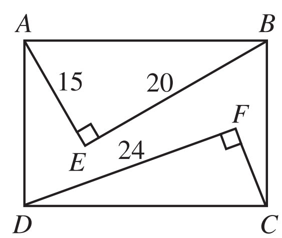
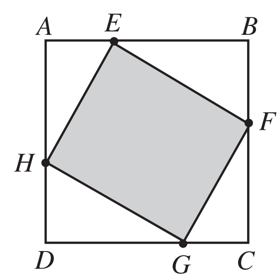
1. (a) In the diagram, what is the area of figure $ABCDEF$?



(b) In the diagram, $ABCD$ is a rectangle with $AE = 15$, $EB = 20$ and $DF = 24$.
What is the length of CF?

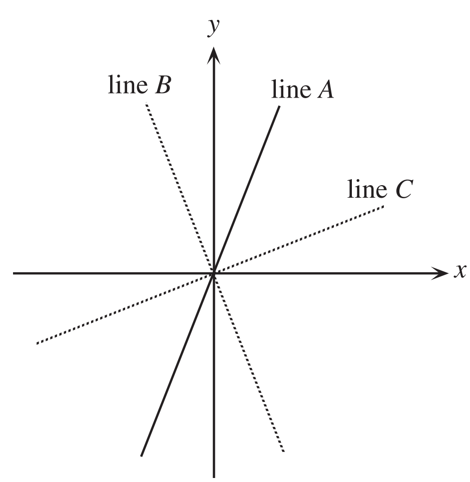


(c) In the diagram, $ABCD$ is a square of side length 6. Points E, F, G, and H are on AB, BC, CD, and DA, respectively, so that the ratios $AE:EB$, $BF:FC$, $CG:GD$, and $DH:HA$ are all equal to 1:2.
What is the area of $EFGH$?

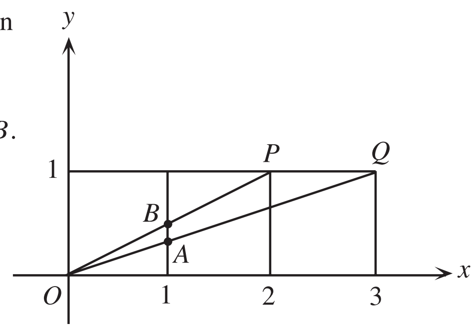


2. (a) A horizontal line has the same y-intercept as the line $3x - y = 6$. What is the equation of this horizontal line?

(b) In the diagram, line A has equation $y = 2x$. Line B is obtained by reflecting line A in the y-axis. Line C is perpendicular to line B.
What is the slope of line C?



(c) Three squares, each of side length 1, are drawn side by side in the first quadrant, as shown. Lines are drawn from the origin to P and Q. Determine, with explanation, the length of AB.



3\. (a) In an arithmetic sequence with five terms, the sum of the first two terms is 2 and the sum of the last two terms is -18. What is the third term of this sequence? (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9, 11 is an arithmetic sequence with five terms.)

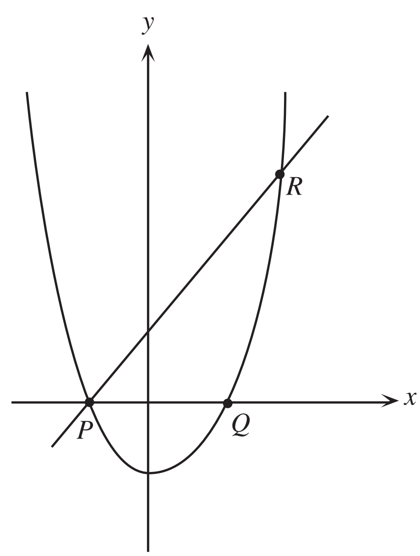
(b) If $x - y = 4\sqrt{2}$ and $xy = 56$, determine the two possible values of $x + y$.

4\. (a) Two fair dice, each having six faces numbered 1 to 6, are thrown. What is the probability that the *product* of the two numbers on the top faces is divisible by 5?

(b) If $f(x) = x^2 - x + 2$, $g(x) = ax + b$, and $f(g(x)) = 9x^2 - 3x + 2$, determine all possible ordered pairs (a, b) which satisfy this relationship.

5\. (a) If $16^x = 2^{x+5} - 2^{x+4}$, determine the value of x.

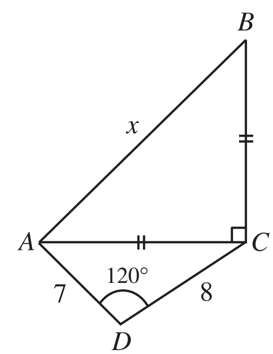
(b) In the diagram, the parabola with equation $y = x^2 + tx - 2$ intersects the x-axis at points P and Q.
Also, the line with equation $y = 3x + 3$ intersects the parabola at points P and R.
Determine the value of t and the area of triangle PQR.



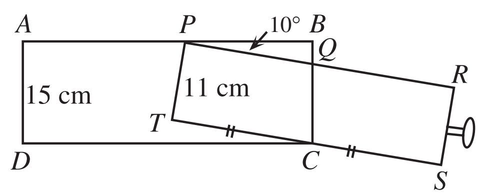
6. (a) Lori has a loonie, three quarters, three dimes, three nickels, and five pennies. She wishes to purchase a toy helicopter for exactly \$1.34. What is the maximum number of coins that she can use to make the purchase? (In Canada, a loonie is worth \$1.00, a quarter is worth \$0.25, a dime is worth \$0.10, a nickel is worth \$0.05, and a penny is worth \$0.01.)

(b) Digital images consist of a very large number of equally spaced dots called *pixels*.
The *resolution* of an image is the number of pixels/cm in each of the horizontal and vertical directions.
Thus, an image with dimensions 10 cm by 15 cm and a resolution of 75 pixels/cm has a total of $(10\times 75)\times(15\times 75)=843\ 750$ pixels.
If each of these dimensions was increased by n % and the resolution was decreased by n %, the image would have 345 600 pixels.
Determine the value of n.

7. (a) In the diagram, $AC=BC$, $AD=7$, $DC=8$, and $\angle ADC=120^\circ$. What is the value of x?



(b) In the diagram, a drawer, $PRST$, that is 11 cm high is in a long slot, $ABCD$, which is 15 cm high. The drawer is pulled out so that the midpoint of its base rests at C. The drawer is tilted so that the top back edge of the drawer touches the top of the slot. If the angle between the drawer and the slot is 10°, determine the length of the drawer, to the nearest tenth of a centimetre.



8. (a) If $T=x^2+\dfrac{1}{x^2}$, determine the values of b and c so that $x^6+\dfrac{1}{x^6}=T^3+bT+c$ for all non-zero real numbers x.

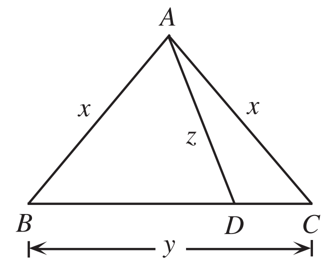
(b) If x is a real number satisfying $x^3+\dfrac{1}{x^3}=2\sqrt{5}$, determine the exact value of $x^2+\dfrac{1}{x^2}$.

9. A *Kirk triplet* is a triple (x, y, z) of integers such that:

   i) $x > z$,

   ii) z is a prime number, and

   iii) there is a triangle ABC with $AB = AC = x$, $BC = y$, and a point D on BC such that $AD = z$ and $\angle ADB = 60°$.



(a) Find the Kirk triplet with $x = 7$ and $z = 5$.

(b) Determine all other Kirk triplets with $z = 5$.

(c) Determine the Kirk triplet for which $\cos(\angle ABC)$ is as close to 0.99 as possible.

10. A *Skolem sequence* of order n is a sequence $(s_1, s_2, ..., s_{2n})$ of $2n$ integers satisfying the conditions:

   i) for every k in $\{1, 2, 3, ..., n\}$, there exist exactly two elements s_i and s_j with $s_i = s_j = k$, and

   ii) if $s_i = s_j = k$ with $i < j$, then $j - i = k$.

For example, $(4, 2, 3, 2, 4, 3, 1, 1)$ is a Skolem sequence of order 4.

(a) List all Skolem sequences of order 4.

(b) Determine, with justification, all Skolem sequences of order 9 which satisfy all of the following three conditions:

   I) $s_3 = 1$,

   II) $s_{18} = 8$, and

   III) between any two equal even integers, there is exactly one odd integer.

(c) Prove that there is no Skolem sequence of order n, if n is of the form $4k + 2$ or $4k + 3$, where k is a non-negative integer.

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Waterloo, ON N2L 3G1

Include your name, address (with postal code), and telephone number.

Cheques or money orders in Canadian funds should be made payable to "Centre for Education in Mathematics and Computing". In Canada, add $3.00 for the first item ordered for shipping and handling, plus $1.00 for each subsequent item. No Provincial Sales Tax is required, but 7% GST must be added. Orders ***outside of Canada ONLY***, add $10.00 for the first item ordered for shipping and handling, plus $2.00 for each subsequent item. **Prices for these publications will remain in effect until September 1, 2004.**

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

Euclid Contest (Grade 12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

Tuesday, April 15, 2003

Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

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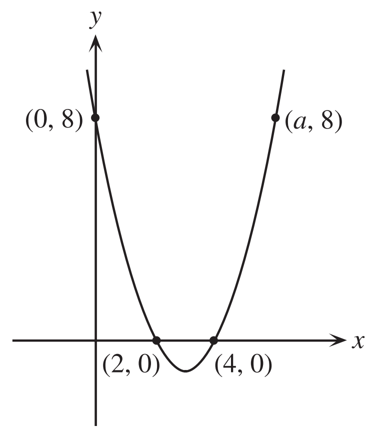
1. (a) In the diagram, the parabola cuts the y-axis at the point $(0, 8)$, cuts the x-axis at the points $(2, 0)$ and $(4, 0)$, and passes through the point $(a, 8)$. What is the value of a?

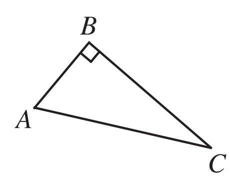


(b) The quadratic equation $x^2 + 6x + k = 0$ has two equal roots. What is the value of k?

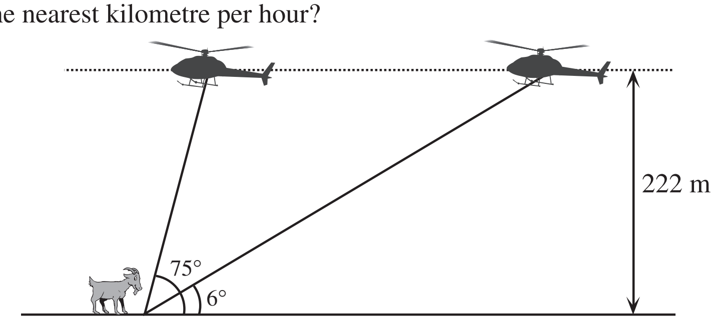
(c) The line $y = 2x + 2$ intersects the parabola $y = x^2 - 3x + c$ at two points. One of these points is $(1, 4)$. Determine the coordinates of the second point of intersection.

2. (a) If $0^\circ < x < 90^\circ$ and $3\sin(x) - \cos(15^\circ) = 0$, what is the value of x to the nearest tenth of a degree?

(b) In the diagram, ΔABC is right-angled at B and $AC = 20$. If $\sin C = \frac{3}{5}$, what is the length of side BC?



(c) A helicopter is flying due west over level ground at a constant altitude of 222 m and at a constant speed. A lazy, stationary goat, which is due west of the helicopter, takes two measurements of the angle between the ground and the helicopter. The first measurement the goat makes is 6° and the second measurement, which he makes 1 minute later, is 75°. If the helicopter has not yet passed over the goat, as shown, how fast is the helicopter travelling to the nearest kilometre per hour?



3. (a) The function $f(x)$ has the property that $f(2x+3)=2f(x)+3$ for all x. If $f(0)=6$, what is the value of $f(9)$?

(b) Suppose that the functions $f(x)$ and $g(x)$ satisfy the system of equations

$$f(x)+3g(x)=x^2+x+6$$
$$2f(x)+4g(x)=2x^2+4$$

for all x. Determine the values of x for which $f(x)=g(x)$.

4. (a) In a short-track speed skating event, there are five finalists including two Canadians. The first three skaters to finish the race win a medal. If all finalists have the same chance of finishing in any position, what is the probability that neither Canadian wins a medal?

(b) Determine the number of positive integers less than or equal to 300 that are multiples of 3 or 5, but are not multiples of 10 or 15.

5. (a) In the series of odd numbers $1+3+5-7-9-11+13+15+17-19-21-23 \ldots$ the signs alternate every three terms, as shown. What is the sum of the first 300 terms of the series?

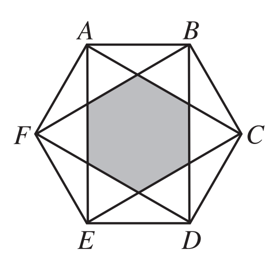
(b) A two-digit number has the property that the square of its tens digit plus ten times its units digit equals the square of its units digit plus ten times its tens digit. Determine all two-digit numbers which have this property, and are prime numbers.

6. (a) A lead box contains samples of two radioactive isotopes of iron. Isotope A decays so that after every 6 minutes, the number of atoms remaining is halved. Initially, there are twice as many atoms of isotope A as of isotope B, and after 24 minutes there are the same number of atoms of each isotope. How long does it take the number of atoms of isotope B to halve?

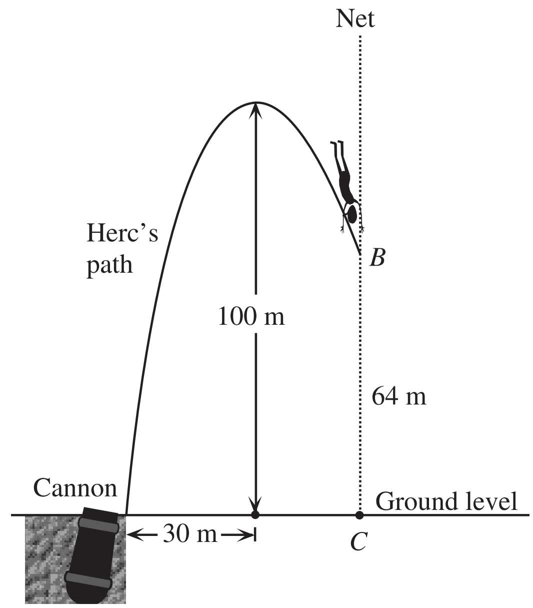
(b) Solve the system of equations:

$$\log_{10}\left(x^3\right)+\log_{10}\left(y^2\right)=11$$
$$\log_{10}\left(x^2\right)-\log_{10}\left(y^3\right)=3$$

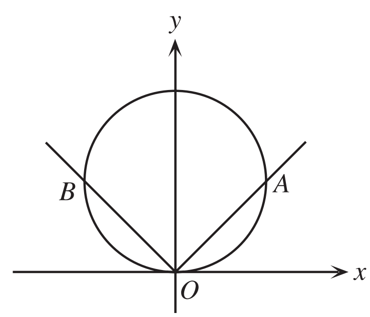
7. (a) A regular hexagon is a six-sided figure which has all of its angles equal and all of its side lengths equal. In the diagram, $ABCDEF$ is a regular hexagon with an area of 36. The region common to the equilateral triangles ACE and BDF is a hexagon, which is shaded as shown. What is the area of the shaded hexagon?



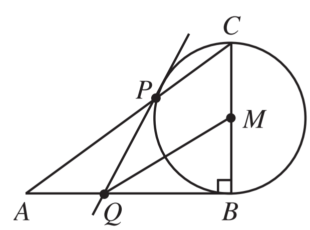
(b) At the Big Top Circus, Herc the Human Cannonball is fired out of the cannon at ground level. (For the safety of the spectators, the cannon is partially buried in the sand floor.) Herc's trajectory is a parabola until he catches the vertical safety net, on his way down, at point B. Point B is 64 m directly above point C on the floor of the tent. If Herc reaches a maximum height of 100 m, directly above a point 30 m from the cannon, determine the horizontal distance from the cannon to the net.



8. (a) A circle with its centre on the y-axis intersects the graph of $y = |x|$ at the origin, O, and exactly two other distinct points, A and B, as shown. Prove that the ratio of the area of triangle ABO to the area of the circle is always $1 : \pi$.



(b) In the diagram, triangle ABC has a right angle at B and M is the midpoint of BC. A circle is drawn using BC as its diameter. P is the point of intersection of the circle with AC. The tangent to the circle at P cuts AB at Q. Prove that QM is parallel to AC.



9. Cyclic quadrilateral $ABCD$ has $AB = AD = 1$, $CD = \cos \angle ABC$, and $\cos \angle BAD = -\frac{1}{3}$. Prove that BC is a diameter of the circumscribed circle.

10. A positive integer n is called "savage" if the integers $\{1, 2, ..., n\}$ can be partitioned into three sets A, B and C such that

i) the sum of the elements in each of A, B, and C is the same,

ii) A contains only odd numbers,

iii) B contains only even numbers, and

iv) C contains every multiple of 3 (and possibly other numbers).

(a) Show that 8 is a savage integer.

(b) Prove that if n is an even savage integer, then $\dfrac{n+4}{12}$ is an integer.

(c) Determine all even savage integers less than 100.

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Euclid Contest (Grade 12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

Tuesday, April 16, 2002

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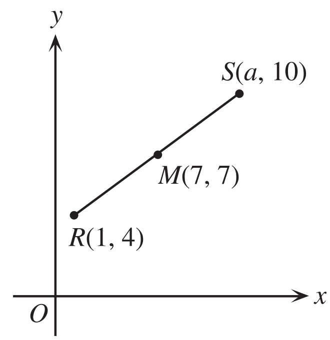
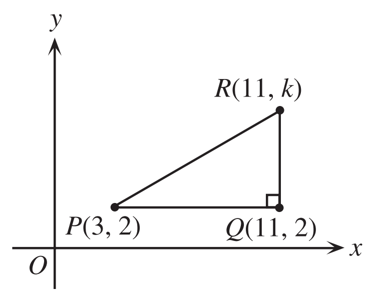


(b) In the diagram, points $P(3,2)$, $Q(11,2)$ and $R(11,k)$ form a triangle with area 24, where $k>0$. What is the value of k?



(c) Lines are *concurrent* if they each pass through the same point. The lines $y=2x+3$, $y=8x+15$, and $y=5x+b$ are concurrent. What is the value of b?

2. (a) The quadratic equation $x^2-3x+c=0$ has $x=4$ as one of its roots. What is its second root?

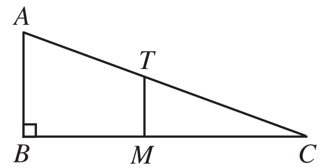
(b) The rational expression $\dfrac{2x^2+1}{x^2-3}$ may be written as $2+\dfrac{A}{x^2-3}$, where A is an integer. What is the value of A?

(c) The parabola $y=x^2-4x+3$ is translated 5 units to the right. In this new position, the equation of the parabola is $y=x^2-14x+d$. Determine the value of d.

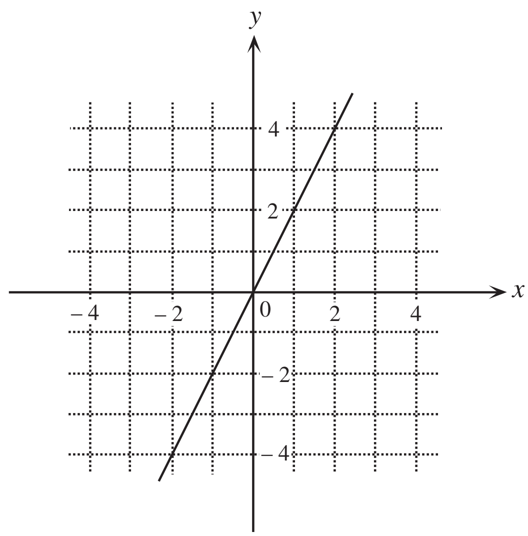
3. (a) Three bins are labelled A, B and C, and each bin contains four balls numbered 1, 2, 3, and 4. The balls in each bin are mixed, and then a student chooses one ball at random from each of the bins. If a, b and c are the numbers on the balls chosen from bins A, B and C, respectively, the student wins a toy helicopter when $a=b+c$. There are 64 ways to choose the three balls. What is the probability that the student wins the prize?

(b) Three positive integers a, ar and ar^2 form an increasing sequence. If the product of the three integers in this sequence is 216, determine all sequences satisfying the given conditions.

4. (a) In the diagram, triangle ABC is right-angled at B. MT is the perpendicular bisector of BC with M on BC and T on AC. If $AT = AB$, what is the size of $\angle ACB$?



(b) The graph of $y = f(x)$, where $f(x) = 2x$, is given on the grid below.



(i) On the grid in the answer booklet, draw and label the graphs of the inverse function $y = f^{-1}(x)$ and the reciprocal function $y = \dfrac{1}{f(x)}$.

(ii) State the coordinates of the points where $f^{-1}(x) = \dfrac{1}{f(x)}$.

(iii) Determine the numerical value of $f^{-1}\left(\dfrac{1}{f\left(\frac{1}{2}\right)}\right)$.

5. (a) What are all values of x such that

$$\log_5(x+3) + \log_5(x-1) = 1?$$

(b) A chef aboard a luxury liner wants to cook a goose. The time t in hours to cook a goose at 180°C depends on the mass of the goose m in kilograms according to the formula

$$t = am^b$$

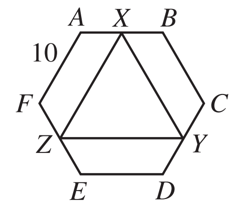
where a and b are constants. The table below gives the times observed to cook a goose at 180°C.

Mass, m (kg)	Time, t (h)
3.00	2.75
6.00	3.75

(i) Using the data in the table, determine both a and b to two decimal places.

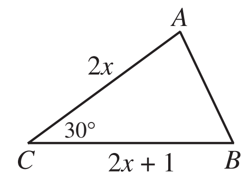
(ii) Suppose that the chef wants to cook a goose with a mass of 8.00 kg at 180°C. How long will it take until his goose is cooked?

6. (a) In the diagram, $ABCDEF$ is a regular hexagon with a side length of 10. If X, Y and Z are the midpoints of AB, CD and EF, respectively, what is the length of XZ?

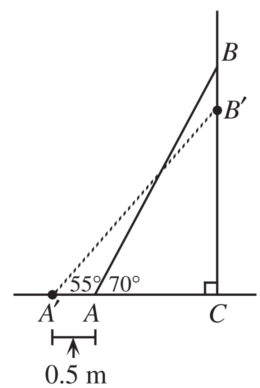


(b) A circle passes through the origin and the points of intersection of the parabolas $y = x^2 - 3$ and $y = -x^2 - 2x + 9$. Determine the coordinates of the centre of this circle.

7. (a) In the diagram, $AC = 2x$, $BC = 2x + 1$ and $\angle ACB = 30°$. If the area of ΔABC is 18, what is the value of x?



(b) A ladder, AB, is positioned so that its bottom sits on horizontal ground and its top rests against a vertical wall, as shown. In this initial position, the ladder makes an angle of 70° with the horizontal. The bottom of the ladder is then pushed 0.5 m away from the wall, moving the ladder to position $A'B'$. In this new position, the ladder makes an angle of 55° with the horizontal. Calculate, to the nearest centimetre, the distance that the ladder slides down the wall (that is, the length of BB').

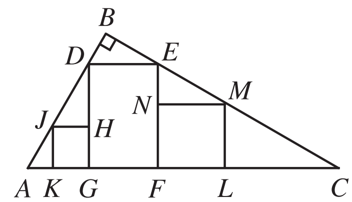


8. (a) In a soccer league with 5 teams, each team plays 20 games (that is, 5 games with each of the other 4 teams). For each team, every game ends in a win (W), a loss (L), or a tie (T). The numbers of wins, losses and ties for each team at the end of the season are shown in the table. Determine the values of x, y and z.

Team	W	L	T
A	2	15	3
B	7	9	4
C	6	12	2
D	10	8	2
E	x	y	z

(b) Prove that it is not possible to create a sequence of 4 numbers a, b, c, d, such that the sum of any two consecutive terms is positive, and the sum of any three consecutive terms is negative.

9. (a) In triangle ABC, $\angle ABC = 90°$. Rectangle $DEFG$ is inscribed in ΔABC, as shown. Squares $JKGH$ and $MLFN$ are inscribed in ΔAGD and ΔCFE, respectively. If the side length of $JHGK$ is v, the side length of $MLFN$ is w, and $DG = u$, prove that $u = v + w$.



(b) Three thin metal rods of lengths 9, 12 and 15 are welded together to form a right-angled triangle, which is held in a horizontal position. A solid sphere of radius 5 rests in the triangle so that it is tangent to each of the three sides. Assuming that the thickness of the rods can be neglected, how high above the plane of the triangle is the top of the sphere?

10. A triangle is called *Heronian* if each of its side lengths is an integer and its area is also an integer. A triangle is called *Pythagorean* if it is right-angled and each of its side lengths is an integer.

(a) Show that every Pythagorean triangle is Heronian.

(b) Show that every odd integer greater than 1 is a side length of some Pythagorean triangle.

(c) Find a Heronian triangle which has all side lengths different, and no side length divisible by 3, 5, 7 or 11.

PUBLICATIONS

Students and parents who enjoy solving problems for fun and recreation may find the following publications of interest. They are an excellent resource for enrichment, problem solving and contest preparation.

COPIES OF PREVIOUS CONTESTS (WITH FULL SOLUTIONS)

Copies of previous contests, together with solutions, are available in a variety of packages, as described below. Please order by package number, and note that the number defines the competition and the number of papers included. Each package has two numbers. Numbers prefixed with E are English language supplies – Numbers prefixed with F are French language supplies. Each package is considered as one title.

(For copies of 1999 - 2001 contests and solutions, please see our website (http://www.cemc.uwaterloo.ca)).

Included is one copy of any one contest, together with solutions, for each of 1998, 1999, and 2000. Recommended for individuals.

E 213, F 213	Gauss Contest (Grades 7, 8)	$10.00
E 513, F 513	Pascal, Cayley, Fermat Contests (Grades 9, 10, 11)	$14.00
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PROBLEMS PROBLEMS PROBLEMS BOOKS

Each volume is a collection of problems (multiple choice and full solution), grouped into 9 or more topics. Questions are selected from previous Canadian Mathematics Competition contests, and full solutions are provided for all questions. The price is $15. **(Available in English only, unless otherwise indicated.)**

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- 10 topics
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- 10 topics (different from Volume 1)
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- 12 topics
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This new book continues the collection of problems available for senior level students. Included for each of the nine chapters is a discussion on solving problems, with suggested approaches. There are more than 160 new problems, almost all from Canadian Mathematics Competitions, with complete solutions. The price is $20. **(Available in English only.)**

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Faculty of Mathematics, Room 5181
University of Waterloo
Waterloo, ON N2L 3G1

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Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest (Grade 12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

Thursday, April 19, 2001

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Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

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NOTE:
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4. It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., except where otherwise indicated.

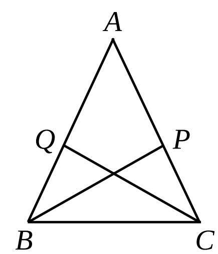
1. (a) What are the values of x such that $(2x-3)^2 = 9$?

   (b) If $f(x) = x^2 - 3x - 5$, what are the values of k such that $f(k) = k$?

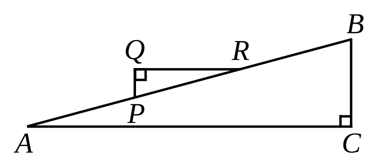
   (c) Determine all (x, y) such that $x^2 + y^2 = 25$ and $x - y = 1$.

2. (a) The vertex of the parabola $y = (x-b)^2 + b + h$ has coordinates $(2, 5)$. What is the value of h?

   (b) In the isosceles triangle ABC, $AB = AC$ and $\angle BAC = 40°$. Point P is on AC such that BP is the bisector of $\angle ABC$. Similarly, Q is on AB such that CQ bisects $\angle ACB$. What is the size of $\angle APB$, in degrees?



   (c) In the diagram, $AB = 300$, $PQ = 20$, and $QR = 100$. Also, QR is parallel to AC. Determine the length of BC, to the nearest integer.

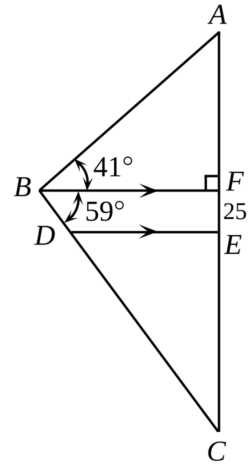


3. (a) In an increasing sequence of numbers with an odd number of terms, the difference between any two consecutive terms is a constant d, and the middle term is 302. When the last 4 terms are removed from the sequence, the middle term of the resulting sequence is 296. What is the value of d?

   (b) There are two increasing sequences of five consecutive integers, each of which have the property that the sum of the squares of the first three integers in the sequence equals the sum of the squares of the last two. Determine these two sequences.

4. (a) If $f(t)=\sin\left(\pi t-\frac{\pi}{2}\right)$, what is the smallest positive value of t at which $f(t)$ attains its minimum value?

(b) In the diagram, $\angle ABF = 41°$, $\angle CBF = 59°$, DE is parallel to BF, and $EF = 25$. If $AE = EC$, determine the length of AE, to 2 decimal places.



5. (a) Determine all integer values of x such that $(x^2-3)(x^2+5)<0$.

(b) At present, the sum of the ages of a husband and wife, P, is six times the sum of the ages of their children, C. Two years ago, the sum of the ages of the husband and wife was ten times the sum of the ages of the same children. Six years from now, it will be three times the sum of the ages of the same children. Determine the number of children.

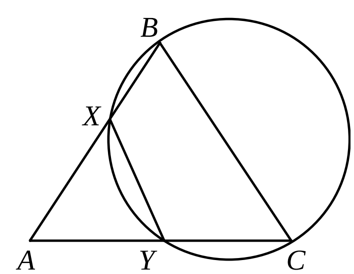
6. (a) Four teams, A, B, C, and D, competed in a field hockey tournament. Three coaches predicted who would win the Gold, Silver and Bronze medals:

Medal	Gold	Silver	Bronze
Team			

- Coach 1 predicted Gold for A, Silver for B, and Bronze for C,
- Coach 2 predicted Gold for B, Silver for C, and Bronze for D,
- Coach 3 predicted Gold for C, Silver for A, and Bronze for D.

Each coach predicted exactly one medal winner correctly. Complete the table **in the answer booklet** to show which team won which medal.

(b) In triangle ABC, $AB = BC = 25$ and $AC = 30$. The circle with diameter BC intersects AB at X and AC at Y. Determine the length of XY.



7. (a) What is the value of x such that $\log_2(\log_2(2x-2))=2$?

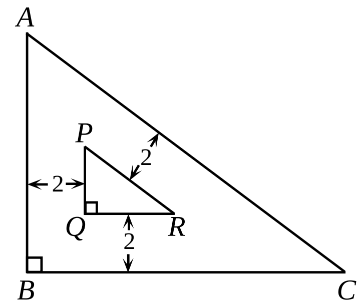
(b) Let $f(x)=2^{kx}+9$, where k is a real number. If $f(3):f(6)=1:3$, determine the value of $f(9)-f(3)$.

8. (a) On the grid provided in the answer booklet, sketch $y = x^2 - 4$ and $y = 2|x|$.

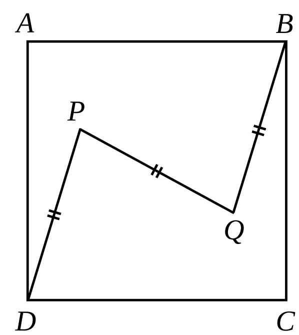
(b) Determine, with justification, all values of k for which $y = x^2 - 4$ and $y = 2|x| + k$ do **not** intersect.

(c) State the values of k for which $y = x^2 - 4$ and $y = 2|x| + k$ intersect in exactly two points. (Justification is not required.)

9. Triangle ABC is right-angled at B and has side lengths which are integers. A second triangle, PQR, is located inside ΔABC as shown, such that its sides are parallel to the sides of ΔABC and the distance between parallel lines is 2. Determine the side lengths of all possible triangles ABC, such that the area of ΔABC is 9 times that of ΔPQR.



10. Points P and Q are located inside the square $ABCD$ such that DP is parallel to QB and $DP = QB = PQ$. Determine the minimum possible value of $\angle ADP$.



Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest (Grade 12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

Tuesday, April 18, 2000

Time: $2\frac{1}{2}$ hours

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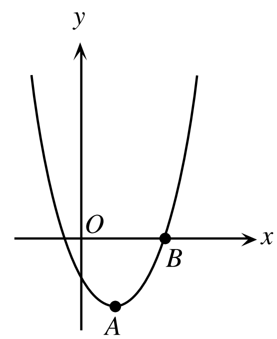
4. It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., except where otherwise indicated.

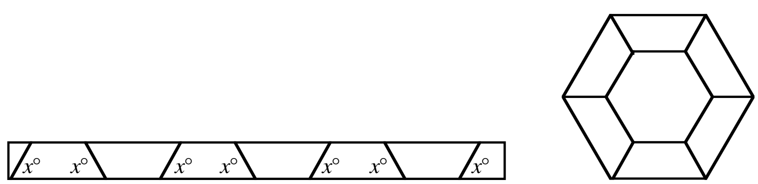
1. (a) If $x+27^{\frac{1}{3}}=125^{\frac{1}{3}}$, what is the value of x?

   (b) The line $y=ax+c$ is parallel to the line $y=2x$ and passes through the point $(1,5)$. What is the value of c?

   (c) The parabola with equation $y=(x-2)^2-16$ has its vertex at A and intersects the x-axis at B, as shown. Determine the equation for the line passing through A and B.



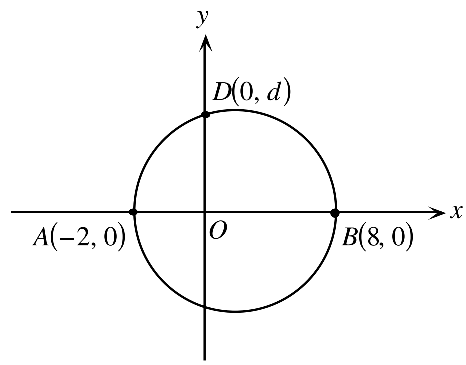
2. (a) Six identical pieces are cut from a board, as shown in the diagram. The angle of each cut is $x°$. The pieces are assembled to form a hexagonal picture frame as shown. What is the value of x?



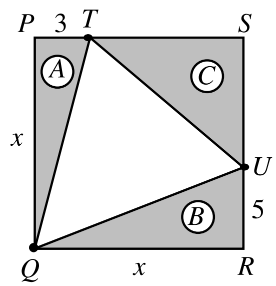
   (b) If $\log_{10} x = 3 + \log_{10} y$, what is the value of $\frac{x}{y}$?

   (c) If $x+\frac{1}{x}=\frac{13}{6}$, determine all values of $x^2+\frac{1}{x^2}$.

3. (a) A circle, with diameter AB as shown, intersects the positive y-axis at point $D(0, d)$. Determine d.



(b) A square $PQRS$ with side of length x is subdivided into four triangular regions as shown so that area Ⓐ + area Ⓑ = area Ⓒ. If $PT = 3$ and $RU = 5$, determine the value of x.



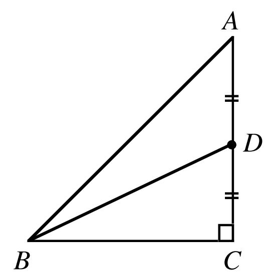
4. (a) A die, with the numbers 1, 2, 3, 4, 6, and 8 on its six faces, is rolled. After this roll, if an odd number appears on the top face, all odd numbers on the die are doubled. If an even number appears on the top face, all the even numbers are halved. If the given die changes in this way, what is the probability that a 2 will appear on the second roll of the die?

(b) The table below gives the final standings for seven of the teams in the English Cricket League in 1998. At the end of the year, each team had played 17 matches and had obtained the total number of points shown in the last column. Each win W, each draw D, each bonus bowling point A, and each bonus batting point B received w, d, a and b points respectively, where w, d, a and b are positive integers. *No points* are given for a loss. Determine the values of w, d, a and b if total points awarded are given by the formula: Points $= w \times W + d \times D + a \times A + b \times B$.

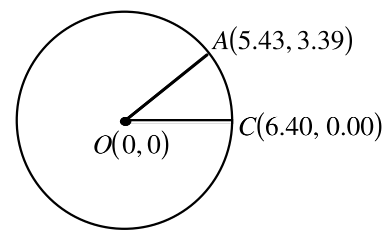
Final Standings

	W	*Losses*	*D*	*A*	*B*	*Points*
Sussex	6	7	4	30	63	201
Warks	6	8	3	35	60	200
Som	6	7	4	30	54	192
Derbys	6	7	4	28	55	191
Kent	5	5	7	18	59	178
Worcs	4	6	7	32	59	176
Glam	4	6	7	36	55	176

5. (a) In the diagram, $AD = DC$, $\sin \angle DBC = 0.6$ and $\angle ACB = 90°$. What is the value of $\tan \angle ABC$?

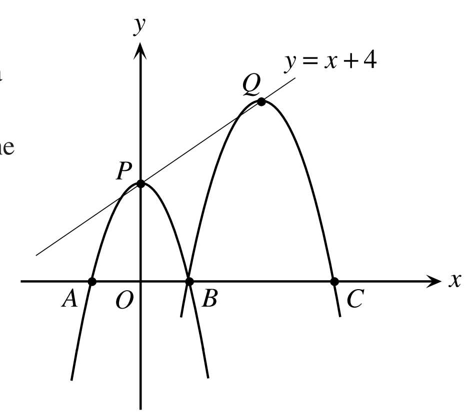


(b) On a cross-sectional diagram of the Earth, the x and y-axes are placed so that $O(0, 0)$ is the centre of the Earth and $C(6.40, 0.00)$ is the location of Cape Canaveral. A space shuttle is forced to land on an island at $A(5.43, 3.39)$, as shown. Each unit represents 1 000 km. Determine the distance from Cape Canaveral to the island, measured on the surface of the earth, to the nearest 10 km.



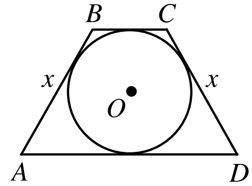
6. (a) Let $\lfloor x \rfloor$ represent the greatest integer which is less than or equal to x. For example, $\lfloor 3 \rfloor = 3$, $\lfloor 2.6 \rfloor = 2$. If x is positive and $x\lfloor x \rfloor = 17$, what is the value of x?

(b) The parabola $y = -x^2 + 4$ has vertex P and intersects the x-axis at A and B. The parabola is translated from its original position so that its vertex moves along the line $y = x + 4$ to the point Q. In this position, the parabola intersects the x-axis at B and C. Determine the coordinates of C.



7. (a) A cube has edges of length n, where n is an integer. Three faces, meeting at a corner, are painted red. The cube is then cut into n^3 smaller cubes of unit length. If exactly 125 of these cubes have no faces painted red, determine the value of n.

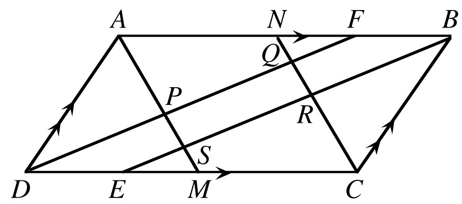
(b) In the isosceles trapezoid $ABCD$, $AB = CD = x$. The area of the trapezoid is 80 and the circle with centre O and radius 4 is tangent to the four sides of the trapezoid. Determine the value of x.



8. In parallelogram $ABCD$, $AB = a$ and $BC = b$, where $a > b$. The points of intersection of the angle bisectors are the vertices of quadrilateral $PQRS$.

(a) Prove that $PQRS$ is a rectangle.

(b) Prove that $PR = a - b$.

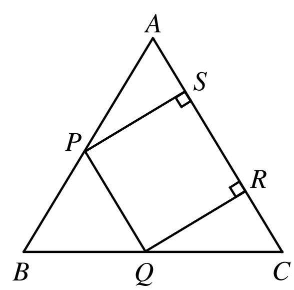


9. A permutation of the integers 1, 2, ..., n is a listing of these integers in some order. For example, $(3,1,2)$ and $(2,1,3)$ are two different permutations of the integers 1, 2, 3. A permutation $(a_1, a_2, ..., a_n)$ of the integers 1, 2, ..., n is said to be "fantastic" if $a_1 + a_2 + ... + a_k$ is divisible by k, for ***each*** k from 1 to n. For example, $(3,1,2)$ is a fantastic permutation of 1, 2, 3 because 3 is divisible by 1, $3+1$ is divisible by 2, and $3+1+2$ is divisible by 3. However, $(2,1,3)$ is not fantastic because $2+1$ is not divisible by 2.

(a) Show that no fantastic permutation exists for $n = 2000$.

(b) Does a fantastic permutation exist for $n = 2001$? Explain.

10. An equilateral triangle ABC has side length 2. A square, $PQRS$, is such that P lies on AB, Q lies on BC, and R and S lie on AC as shown. The points P, Q, R, and S move so that P, Q and R always remain on the sides of the triangle and S moves from AC to AB through the interior of the triangle. If the points P, Q, R and S always form the vertices of a square, show that the path traced out by S is a straight line parallel to BC.



Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest (Grade 12)

for the

NATIONAL BANK OF CANADA

Awards

Tuesday, April 20, 1999

Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

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Instructions for SHORT ANSWER parts

1. **SHORT ANSWER** parts are indicated like this:
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Instructions for FULL SOLUTION parts

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NOTE: At the completion of the contest, insert the information sheet inside the answer booklet.

NOTE: 1. Please read the instructions on the front cover of this booklet.
2. Place all answers in the answer booklet provided.
3. For questions marked " ", full marks will be given for a correct answer which is to be placed in the appropriate box in the answer booklet. **Part marks will be given for work shown**. Students are strongly encouraged to show their work.
4. It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., except where otherwise indicated.

1. (a) If $x^{-1} = 3^{-1} + 4^{-1}$, what is the value of x?

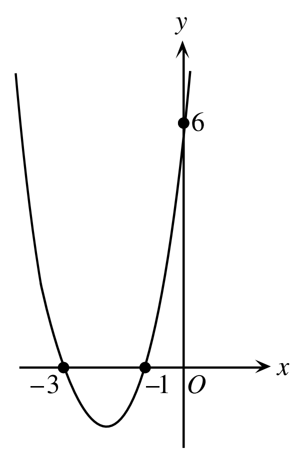
(b) If the point $P(-3, 2)$ is on the line $3x + 7ky = 5$, what is the value of k?

(c) If $x^2 - x - 2 = 0$, determine all possible values of $1 - \frac{1}{x} - \frac{6}{x^2}$.

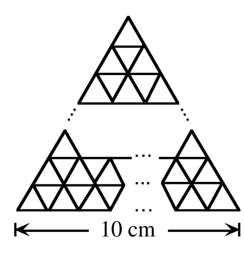
2. (a) The circle defined by the equation $(x+4)^2 + (y-3)^2 = 9$ is moved horizontally until its centre is on the line $x = 6$. How far does the centre of the circle move?

(b) The parabola defined by the equation $y = (x-1)^2 - 4$ intersects the x-axis at the points P and Q. If (a, b) is the mid-point of the line segment PQ, what is the value of a?

(c) Determine an equation of the quadratic function shown in the diagram.

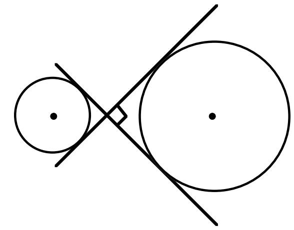


3. (a) How many equilateral triangles of side 1 cm, placed as shown in the diagram, are needed to completely cover the interior of an equilateral triangle of side 10 cm?



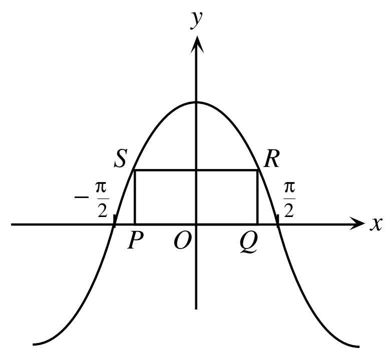
(b) The populations of Alphaville and Betaville were equal at the end of 1995. The population of Alphaville decreased by 2.9% during 1996, then increased by 8.9% during 1997, and then increased by 6.9% during 1998. The population of Betaville increased by r% in each of the three years. If the populations of the towns are equal at the end of 1998, determine the value of r correct to one decimal place.

4. (a) In the diagram, the tangents to the two circles intersect at 90° as shown. If the radius of the smaller circle is 2, and the radius of the larger circle is 5, what is the distance between the centres of the two circles?

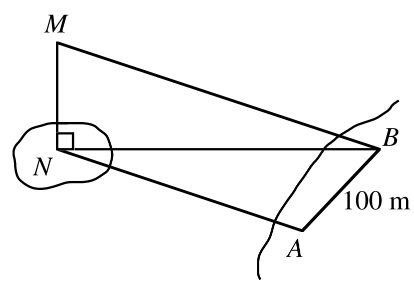


(b) A circular ferris wheel has a radius of 8 m and rotates at a rate of 12° per second. At $t = 0$, a seat is at its lowest point which is 2 m above the ground. Determine how high the seat is above the ground at $t = 40$ seconds.

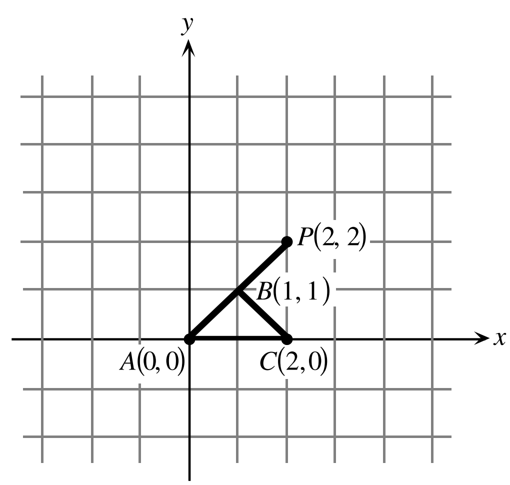
5. (a) A rectangle $PQRS$ has side PQ on the x-axis and touches the graph of $y = k \cos x$ at the points S and R as shown. If the length of PQ is $\frac{\pi}{3}$ and the area of the rectangle is $\frac{5\pi}{3}$, what is the value of k?



(b) In determining the height, MN, of a tower on an island, two points A and B, 100 m apart, are chosen on the same horizontal plane as N. If $\angle NAB = 108°$, $\angle ABN = 47°$ and $\angle MBN = 32°$, determine the height of the tower to the nearest metre.



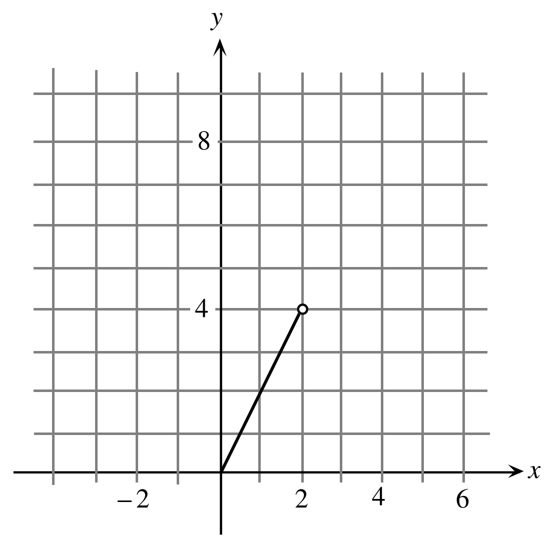
6. (a) The points A, P and a third point Q (not shown) are the vertices of a triangle which is similar to triangle ABC. What are the coordinates of all possible positions for Q?



(b) Determine the coordinates of the points of intersection of the graphs of $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$.

7. (a) On the grid provided in the answer booklet, draw the graphs of the functions $y=-2\sqrt{x+1}$ and $y=\sqrt{x-2}$. For what value(s) of k will the graphs of the functions $y=-2\sqrt{x+1}$ and $y=\sqrt{x-2}+k$ intersect? (Assume x and k are real numbers.)

(b) Part of the graph for $y=f(x)$ is shown, $0 \le x < 2$.
If $f(x+2)=\frac{1}{2}f(x)$ for all real values of x, draw the graph for the intervals, $-2 \le x < 0$ and $2 \le x < 6$.

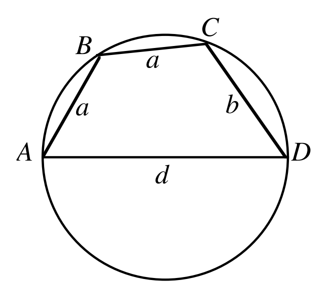


8. (a) The equation $y=x^2+2ax+a$ represents a parabola for all real values of a. Prove that each of these parabolas pass through a common point and determine the coordinates of this point.

(b) The vertices of the parabolas in part (a) lie on a curve. Prove that this curve is itself a parabola whose vertex is the common point found in part (a).

9. A 'millennium' series is any series of consecutive integers with a sum of 2000. Let m represent the first term of a 'millennium' series.
(a) Determine the minimum value of m.
(b) Determine the smallest possible positive value of m.

10. $ABCD$ is a cyclic quadrilateral, as shown, with side $AD=d$, where d is the diameter of the circle. $AB=a$, $BC=a$ and $CD=b$. If a, b and d are integers $a \ne b$,
(a) prove that d cannot be a prime number.
(b) determine the *minimum* value of d.



Canadian Mathematics Competition

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1. (a) If one root of $x^2+2x-c=0$ is $x=1$, what is the value of c?

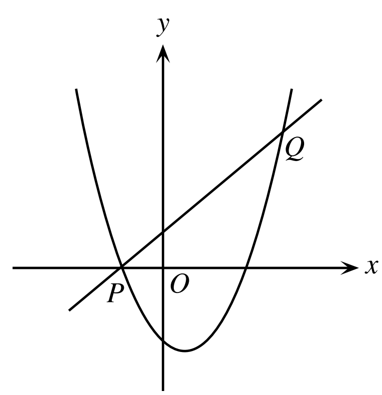
(b) If $2^{2x-4}=8$, what is the value of x?

(c) Two perpendicular lines with x-intercepts -2 and 8 intersect at $(0, b)$. Determine all values of b.

2. (a) The vertex of $y=(x-1)^2+b$ has coordinates $(1, 3)$. What is the y-intercept of this parabola?

(b) What is the area of ΔABC with vertices $A(-3, 1)$, $B(5, 1)$ and $C(8, 7)$?

(c) In the diagram, the line $y=x+1$ intersects the parabola $y=x^2-3x-4$ at the points P and Q. Determine the coordinates of P and Q.



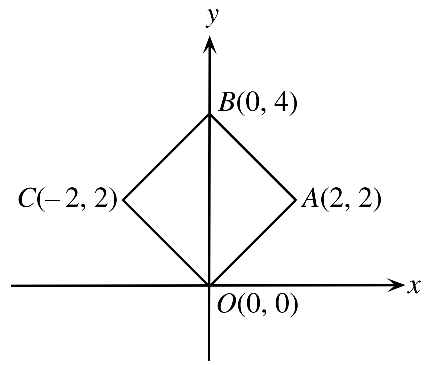
3. (a) The graph of $y=m^x$ passes through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

(b) Jane bought 100 shares of stock at \$10.00 per share. When the shares increased to a value of \$$N$ each, she made a charitable donation of all the shares to the Euclid Foundation. She received a tax refund of 60% on the total value of her donation. However, she had to pay a tax of 20% on the increase in the value of the stock. Determine the value of N if the difference between her tax refund and the tax paid was \$1000.

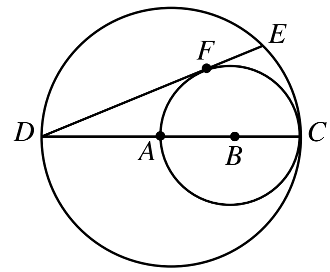
4. (a) Consider the sequence $t_1=1$, $t_2=-1$ and $t_n=\left(\dfrac{n-3}{n-1}\right)t_{n-2}$ where $n\geq 3$. What is the value of t_{1998}?

(b) The nth term of an arithmetic sequence is given by $t_n=555-7n$.
If $S_n=t_1+t_2+...+t_n$, determine the smallest value of n for which $S_n<0$.

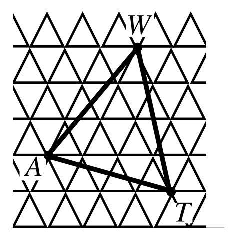
5. (a) A square $OABC$ is drawn with vertices as shown. Find the equation of the circle with largest area that can be drawn inside the square.



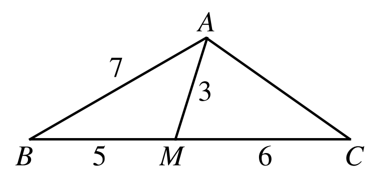
(b) In the diagram, DC is a diameter of the larger circle centred at A, and AC is a diameter of the smaller circle centred at B. If DE is tangent to the smaller circle at F, and $DC = 12$, determine the length of DE.



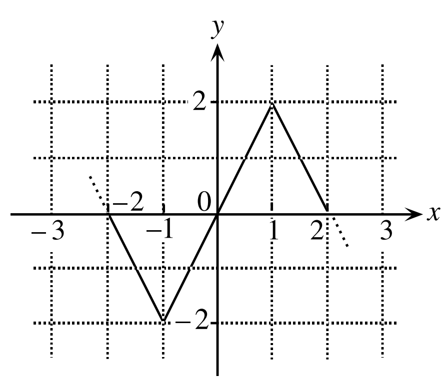
6. (a) In the grid, each small equilateral triangle has side length 1. If the vertices of ΔWAT are themselves vertices of small equilateral triangles, what is the area of ΔWAT?



(b) In ΔABC, M is a point on BC such that $BM = 5$ and $MC = 6$. If $AM = 3$ and $AB = 7$, determine the exact value of AC.



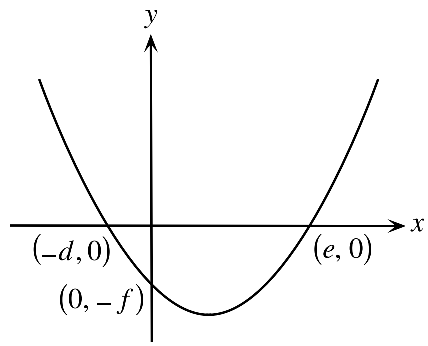
7. (a) The function $f(x)$ has period 4. The graph of one period of $y = f(x)$ is shown in the diagram. Sketch the graph of $y = \frac{1}{2}[f(x-1) + f(x+3)]$, for $-2 \le x \le 2$.



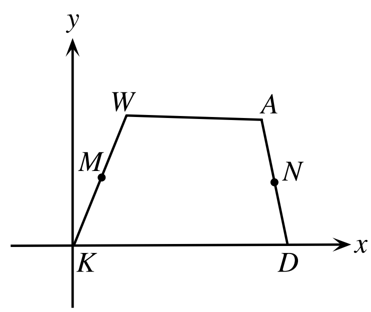
(b) If x and y are real numbers, determine all solutions (x, y) of the system of equations

$$x^2 - xy + 8 = 0$$
$$x^2 - 8x + y = 0.$$

8. (a) In the graph, the parabola $y = x^2$ has been translated to the position shown.
Prove that $de = f$.

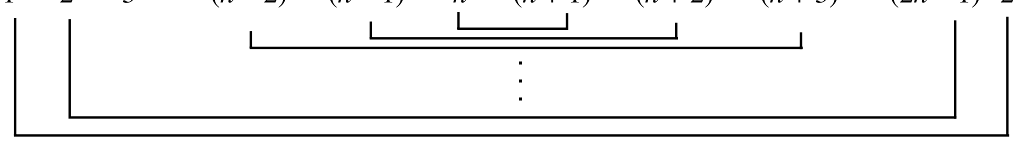


(b) In quadrilateral $KWAD$, the midpoints of KW and AD are M and N respectively. If $MN = \frac{1}{2}(AW + DK)$, prove that WA is parallel to KD.



9. Consider the first $2n$ natural numbers. Pair off the numbers, as shown, and multiply the two members of each pair. Prove that there is no value of n for which two of the n products are equal.

$$1 \quad 2 \quad 3 \ \cdots \ (n-2) \quad (n-1) \quad n \quad (n+1) \quad (n+2) \quad (n+3) \ \cdots \ (2n-1) \quad 2n$$



10. The equations $x^2 + 5x + 6 = 0$ and $x^2 + 5x - 6 = 0$ ***each*** have integer solutions whereas only one of the equations in the pair $x^2 + 4x + 5 = 0$ and $x^2 + 4x - 5 = 0$ has integer solutions.

(a) Show that if $x^2 + px + q = 0$ and $x^2 + px - q = 0$ ***both*** have integer solutions, then it is possible to find integers a and b such that $p^2 = a^2 + b^2$. (i.e. (a, b, p) is a Pythagorean triple).

(b) Determine q in terms of a and b.