The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Euclid Contest

Wednesday, April 3, 2024
(in North America and South America)

Thursday, April 4, 2024
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

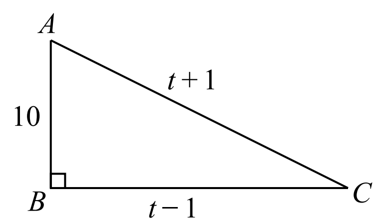
NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked (lightbulb), place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked (pencil), provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions, and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about residency.

1. (a) If $x = 2$, what is the value of $\dfrac{x^4 + 3x^2}{x^2}$?

   (b) In the diagram, $\triangle ABC$ is right-angled at B. Also, $AB = 10$, $BC = t - 1$, and $AC = t + 1$. What is the value of t?



   (c) Suppose that $\dfrac{2}{y} + \dfrac{3}{2y} = 14$. Determine the value of y.

2. (a) In a sequence with six terms, each term after the second is the sum of the previous two terms. If the fourth term is 13 and the sixth term is 36, what is the first term?

   (b) For some real number $r \neq 0$, the sequence $5r$, $5r^2$, $5r^3$ has the property that the second term plus the third term equals the square of the first term. What is the value of r?

   (c) Jimmy wrote four tests last week. The average of his marks on the first, second and third tests was 65. The average of his marks on the second, third and fourth tests was 80. His mark on the fourth test was 2 times his mark on the first test. Determine his mark on the fourth test.

3. (a) The graph of the equation $y = r(x-3)(x-r)$ intersects the y-axis at $(0, 48)$. What are the two possible values of r?

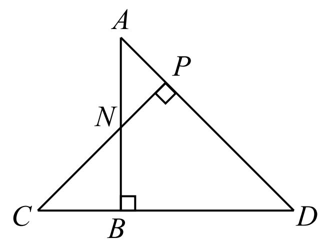
(b) A bicycle costs \$$B$ before taxes. If the sales tax were 13%, Annemiek would pay a total that is \$24 higher than if the sales tax were 5%. What is the value of B?

(c) The function f has the following three properties:

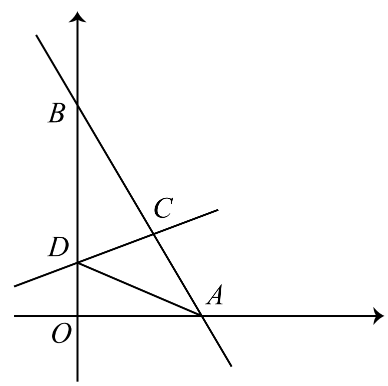
- $f(1) = 3$.
- $f(2n) = (f(n))^2$ for all positive integers n.
- $f(2m+1) = 3f(2m)$ for all positive integers m.

Determine the value of $f(2) + f(3) + f(4)$.

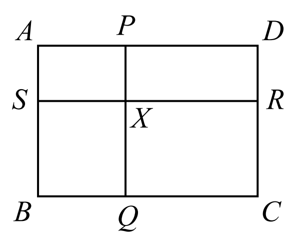
4. (a) In the diagram, AB is perpendicular to CD, CP is perpendicular to AD, and N is the point of intersection of AB and CP. Also, $\angle ADB = 45^\circ$, $AB = 12$, and $CB = 6$. What is the area of $\triangle APN$?



(b) In the diagram, the line with equation $y = -3x + 6$ crosses the x-axis at A and the y-axis at B. Suppose that $m > 0$ and that the line with equation $y = mx + 1$ crosses the y-axis at D and intersects the line with equation $y = -3x + 6$ at the point C. If O is the origin and the area of $\triangle ACD$ is $\frac{1}{2}$ of the area of $\triangle ABO$, determine the coordinates of C.



5. (a) In the diagram, rectangle $ABCD$ is divided into four smaller rectangles by the lines PQ and RS, which intersect at X. The areas of these smaller rectangles are, in some order, 2, 6, 3, and a. What are the three possible values of a?



(b) Suppose that the parabola with equation $y = x^2 - 4tx + 5t^2 - 6t$ has two distinct x-intercepts. Determine the value of t for which the distance between these x-intercepts is as large as possible.

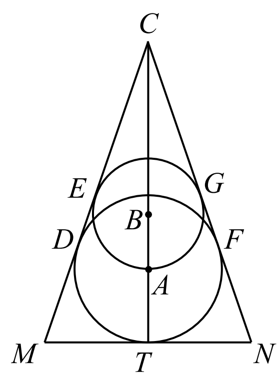
6. (a) There are M integers between 10 000 and 100 000 that are multiples of 21 and whose units (ones) digit is 1. What is the value of M?

(b) There are N students who attend Strickland S.S., where $500 < N < 600$. Among these N students, $\frac{2}{5}$ are in the physics club and $\frac{1}{4}$ are in the math club. In the physics club, there are 2 times as many students who are not in the math club as there are students who are in the math club. Determine the number of students who are not in either club.

7. (a) Arun and Bella run around a circular track, starting from diametrically opposite points. Arun runs clockwise around the track and Bella runs counterclockwise. Arun and Bella run at constant, but different, speeds. They meet for the first time after Arun has run 100 m. They meet for the second time after Bella runs 150 m past their first meeting point. What is the length of the track?

(b) Determine all angles θ with $0^\circ \leq \theta \leq 360^\circ$ for which $4^{1+\cos^3\theta} = 2^{2-\cos\theta} \cdot 8^{\cos^2\theta}$.

8. (a) In the diagram, the circle with centre A has radius 4, and A lies on the circle which has centre B and radius 3. The line passing through A and B lies along the diameter of each circle and is perpendicular to MN at T. Also, MN is tangent to the larger circle, and MC and NC are each tangent to both circles at points D, E, F, and G, as shown. Determine the area of $\triangle MNC$.



(b) Determine all triples (x, y, z) of real numbers that are solutions to the following system of equations:

$$\begin{aligned} \log_9 x + \log_9 y + \log_3 z &= 2 \\ \log_{16} x + \log_4 y + \log_{16} z &= 1 \\ \log_5 x + \log_{25} y + \log_{25} z &= 0 \end{aligned}$$

9. An ant walks along the x-axis by taking a sequence of steps of length 1. Some, all or none of these steps are in the positive x-direction; some, all or none of these steps are in the negative x-direction. The ant begins at $x = 0$, takes a total of n steps, and ends at $x = d$. For each such sequence, let c be the number of times that the ant changes direction.

(a) Determine the number of different sequences of steps for which $n = 9$ and $d = 5$.

(b) Suppose that $n = 9$ and $d = 3$. Determine the number of sequences for which c is even.

(c) Determine the number of pairs (d, n) of integers with $1 \leq n \leq 2024$ and $d \geq 0$ for which c is even for exactly half of the sequences of n steps that end at $x = d$.

10. Suppose that s and t are real numbers with $0 < s \leq 1$ and $0 < t \leq 1$. Points $A(-1, 0)$, $B(0, 4)$ and $C(1, 0)$ form $\triangle ABC$. Points $S(s, 0)$ and $T(-t, 0)$ lie on AC. Point P lies on AB and point Q lies on BC, with neither P nor Q at a vertex of $\triangle ABC$. Line segments SP and TQ intersect at X and partition $\triangle ABC$ into four regions. For some such pairs (s, t) of real numbers and points P and Q, the line segments SP and TQ in fact partition $\triangle ABC$ into four regions of equal area. We call such a pair (s, t) a *balancing* pair.

(a) Suppose that (s, t) is a balancing pair with $s = 1$ and that line segments SP and TQ partition $\triangle ABC$ into four regions of equal area. Determine the coordinates of P.

(b) Prove that there exist real numbers d, e, f, and g for which all balancing pairs (s, t) satisfy an equation of the form

$$s^2 + t^2 = dst + es + ft + g$$

and determine the values of d, e, f, and g.

(c) Determine an infinite family of distinct pairs of rational numbers (s, t) with $0 < s \leq t \leq 1$ that satisfy the equation in (b).

2024 Euclid Contest (English)

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For students...

Thank you for writing the 2024 Euclid Contest! Each year, more than 260 000 students from more than 80 countries register to write the CEMC's Contests.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2024 Canadian Senior Mathematics Contest, which will be written in November 2024.

Visit our website `cemc.uwaterloo.ca` to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

For teachers...

Visit our website `cemc.uwaterloo.ca` to

- Obtain information about our 2024/2025 contests
- Look at our free online courseware for high school students
- Use our free Problem Set Generator to create problem sets for curriculum support and enrichment
- Learn about our face-to-face workshops and our web resources
- Subscribe to our free Problem of the Week
- Investigate our online Master of Mathematics for Teachers
- Find your school's contest results