The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2023 Pascal Contest

(Grade 9)

Wednesday, February 22, 2023
(in North America and South America)

Thursday, February 23, 2023
(outside of North America and South America)

Solutions

1. Since 110 003 is greater than 110 000 and each of the other four choices is less than 110 000, the integer 110 003 is the greatest of all of the choices.

   ANSWER: (B)

2. From left to right, the number of shaded squares in each column with shaded squares is 1, 3, 5, 4, 2.
   Thus, the number of shaded squares is $1 + 3 + 5 + 4 + 2 = 15$.
   Alternatively, we could note that exactly one-half of the 30 squares are shaded since each column with shaded squares can be paired with a column of the same number of unshaded squares. (The 1st column is paired with the 8th, the 2nd with the 7th, the 3rd with the 6th, and the 4th with the 5th.) Thus, again there are $\frac{1}{2} \times 30 = 15$ shaded squares.

   ANSWER: (C)

3. Evaluating, $2^3 - 2 + 3 = 2 \times 2 \times 2 - 2 + 3 = 8 - 2 + 3 = 9$.

   ANSWER: (C)

4. Since $3 + \triangle = 5$, then $\triangle = 5 - 3 = 2$.
   Since $\triangle + \square = 7$ and $\triangle = 2$, then $\square = 5$.
   Thus, $\triangle + \triangle + \triangle + \square + \square = 3 \times 2 + 2 \times 5 = 6 + 10 = 16$.

   ANSWER: (E)

5. Evaluating, $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} = 0.3 + 0.03 + 0.003 = 0.333$.

   ANSWER: (A)

6. Since $\frac{1}{3}$ of x is equal to 4, then x is equal to 3×4 or 12. Thus, $\frac{1}{6}$ of x is equal to $12 \div 6 = 2$.
   Alternatively, since $\frac{1}{6}$ is one-half of $\frac{1}{3}$, then $\frac{1}{6}$ of x is equal to one-half of $\frac{1}{3}$ of x, which is $4 \div 2$ or 2.

   ANSWER: (C)

7. Jurgen takes $25 + 35 = 60$ minutes to pack and then walk to the bus station.
   Since Jurgen arrives 60 minutes before the bus leaves, he began packing $60+60 = 120$ minutes, or 2 hours, before the bus leaves.
   Since the bus leaves at 6:45 p.m., Jurgen began packing at 4:45 p.m.

   ANSWER: (A)

8. Since the letters of RHOMBUS take up 7 of the 31 spaces on the line, there are $31 - 7 = 24$ spaces that are empty.
   Since the numbers of empty spaces on each side of RHOMBUS are the same, there are $24 \div 2 = 12$ empty spaces on each side.
   Therefore, the letter R is placed in space number $12 + 1 = 13$, counting from the left.

   ANSWER: (B)

9. The digits to the right of the decimal place in the decimal represenation of $\frac{1}{7}$ occur in blocks of 6, repeating the block of digits 142857.
   Since $16 \times 6 = 96$, then the 96th digit to the right of the decimal place is the last in one of these blocks; that is, the 96th digit is 7.
   This means that the 97th digit is 1, the 98th digit is 4, the 99th digit is 2, and the 100th digit is 8.

   ANSWER: (D)

10. The path that the ant walks from A to B is vertical and has length 5.
The path that the ant walks from B to C is horizontal and has length 8.
The path that the ant walks from C to A does not follow the gridlines. To determine the length of CA, we can use the Pythagorean Theorem because AB and BC meet at a right angle.
This gives $CA^2 = AB^2 + BC^2 = 5^2 + 8^2 = 25 + 64 = 89$.
Since $CA > 0$, then $CA = \sqrt{89}$.
Thus, the total distance that the ant walks is $5 + 8 + \sqrt{89}$ or $13 + \sqrt{89}$.
Answer: (D)

11. Suppose that the original prism has length ℓ cm, width w cm, and height h cm.
Since the volume of this prism is 12 cm^3, then $\ell wh = 12$.
The new prism has length 2ℓ cm, width $2w$ cm, and height $3h$ cm.
The volume of this prism, in cm^3, is $(2l) \times (2w) \times (3h) = 2 \times 2 \times 3 \times lwh = 12 \times 12 = 144$.
Answer: (E)

12. Since $31 = 3 \times 10 + 1$ and $94 = 3 \times 31 + 1$ and $331 = 3 \times 110 + 1$ and $907 = 3 \times 302 + 1$, then each of 31, 94, 331, and 907 appear in the second column of Morgan's spreadsheet.
Thus, 131 must be the integer that does not appear in Morgan's spreadsheet. (We note that 131 is 2 more than $3 \times 43 = 129$ so is not 1 more than a muliple of 3.)
Answer: (C)

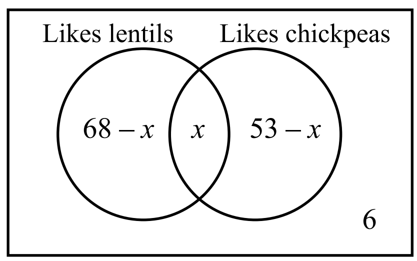
13. The total decrease in temperature between these times is $16.2^\circ\text{C} - (-3.6^\circ\text{C}) = 19.8^\circ\text{C}$.
The length of time between 3:00 p.m. one day and 2:00 a.m. the next day is 11 hours, since it is 1 hour shorter than the length of time between 3:00 p.m. and 3:00 a.m.
Since the temperature decreased at a constant rate over this period of time, the rate of decrease in temperature was $\dfrac{19.8^\circ\text{C}}{11\text{ h}} = 1.8^\circ\text{C/h}$.
Answer: (B)

14. There are 2 possible "states" for each door: open or closed.
Therefore, there are $2 \times 2 \times 2 \times 2 = 2^4 = 16$ possible combinations of open and closed for the 4 doors.
If exactly 2 of the 4 doors are open, these doors could be the 1st and 2nd, or 1st and 3rd, or 1st and 4th, or 2nd and 3rd, or 2nd and 4th, or 3rd and 4th. Thus, there are 6 ways in which 2 of the 4 doors can be open.
Since each door is randomly open or closed, then the probability that exactly 2 doors are open is $\frac{6}{16}$ which is equivalent to $\frac{3}{8}$.
Answer: (A)

15. Nasim can buy 24 cards by buying three 8-packs ($3 \times 8 = 24$).
Nasim can buy 25 cards by buying five 5-packs ($5 \times 5 = 25$).
Nasim can buy 26 cards by buying two 5-packs and two 8-packs ($2 \times 5 + 2 \times 8 = 26$).
Nasim can buy 28 cards by buying four 5-packs and one 8-pack ($4 \times 5 + 1 \times 8 = 28$).
Nasim can buy 29 cards by buying one 5-pack and three 8-packs ($1 \times 5 + 3 \times 8 = 29$).
Nasim cannot buy exactly 27 cards, because the number of cards in 8-packs that he buys would be 0, 8, 16, or 24, leaving 27, 19, 11, or 3 cards to buy in 5-packs. None of these are possible, since none of 27, 19, 11, or 3 is a multiple of 5.
Therefore, for 5 of the 6 values of n, Nasim can buy exactly n cards.
Answer: (A)

16. Suppose that Mathilde had m coins at the start of last month and Salah had s coins at the start of last month.
From the given information, 100 is 25% more than m, so $100 = 1.25m$ which means that $m = \dfrac{100}{1.25} = 80$.
From the given information, 100 is 20% less than s, so $100 = 0.80s$ which means that $s = \dfrac{100}{0.80} = 125$.
Therefore, at the beginning of last month, they had a total of $m + s = 80 + 125 = 205$ coins.
ANSWER: (E)

17. Suppose that x students like both lentils and chickpeas.
Since 68 students like lentils, these 68 students either like chickpeas or they do not.
Since x students like lentils and chickpeas, then x of the 68 students that like lentils also like chickpeas and so $68 - x$ students like lentils but do not like chickpeas.
Since 53 students like chickpeas, then $53 - x$ students like chickpeas but do not like lentils.
We know that there are 100 students in total and that 6 like neither lentils nor chickpeas.
We use a Venn diagram to summarize this information:



Since there are 100 students in total, then $(68 - x) + x + (53 - x) + 6 = 100$ which gives $127 - x = 100$ and so $x = 27$.
Therefore, there are 27 students that like both lentils and chickpeas.
ANSWER: (B)

18. Since $\angle ABD = 180^\circ$ and $\angle ABC = x^\circ$, then $\angle CBD = 180^\circ - x^\circ$.
Since the measures of the angles in $\triangle BCD$ add to 180°, then

$$\angle BDC = 180^\circ - (180^\circ - x^\circ) - 90^\circ = x^\circ - 90^\circ$$

Similarly, $\angle GFD = 180^\circ$ and $\angle FDE = y^\circ - 90^\circ$. Finally, $\angle BDF = 180^\circ$ and so

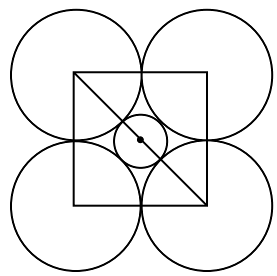
$$\begin{aligned} \angle BDC + \angle CDE + \angle FDE &= 180^\circ \\ (x^\circ - 90^\circ) + 80^\circ + (y^\circ - 90^\circ) &= 180^\circ \\ x + y - 100 &= 180 \end{aligned}$$

and so $x + y = 280$.
ANSWER: (D)

19. Before Kyne removes hair clips, Ellie has 4 red clips and $4 + 5 + 7 = 16$ clips in total, so the probability that she randomly chooses a red clip is $\frac{4}{16}$ which equals $\frac{1}{4}$.
After Kyne removes the clips, the probability that Ellie chooses a red clip is $2 \times \frac{1}{4}$ or $\frac{1}{2}$.
Since Ellie starts with 4 red clips, then after Kyne removes some clips, Ellie must have 4, 3, 2, 1, or 0 red clips.
Since the probability that Ellie chooses a red clip is larger than 0, she cannot have 0 red clips.
Since the probability of her choosing a red clip is $\frac{1}{2}$, then the total number of clips that she has after k are removed must be twice the number of red clips, so could be 8, 6, 4, or 2.
Thus, the possible values of k are $16 - 8 = 8$ or $16 - 6 = 10$ or $16 - 4 = 12$ or $16 - 2 = 14$.
Of these, 12 is one of the given possibilities. (One possibility is that Kyne removes 2 of the red clips, 5 of the blue clips and 5 of the green clips, leaving 2 red clips and 2 green clips.)

Answer: (C)

20. Draw one of the diagonals of the square. The diagonal passes through the centre of the square.



By symmetry, the centre of the smaller circle is the centre of the square. (If it were not the centre of the square, then one of the four larger circles would have to be different from the others somehow, which is not true.)
Further, the diagonals of the square pass through the points where the smaller circle is tangent to the larger circles. (The line segment from each vertex of the square to the centre of the smaller circle passes through the point of tangency. These four segments are equal in length and meet at right angles since the diagram can be rotated by 90 degrees without changing its appearance. Thus, each of these is half of a diagonal.)
Since each of the larger circles has radius 5, the side length of the square is $5 + 5 = 10$.
Since the square has side length 10, its diagonal has length $\sqrt{10^2 + 10^2} = \sqrt{200}$ by the Pythagorean Theorem.
Therefore, $5 + 2r + 5 = \sqrt{200}$ which gives $2r = \sqrt{200} - 10$ and so $r \approx 2.07$.
Of the given choices, r is closest to 2.1, or (C).

Answer: (C)

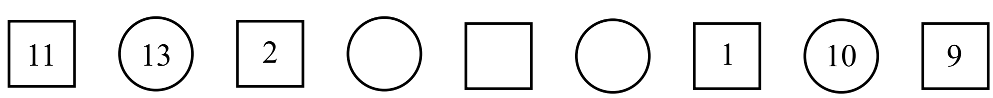
21. We follow Alicia's algorithm carefully:

- Step 1: Alicia writes down $m = 3$ as the first term.
- Step 2: Since $m = 3$ is odd, Alicia sets $n = m + 1 = 4$.
- Step 3: Alicia writes down $m + n + 1 = 8$ as the second term.
- Step 4: Alicia sets $m = 8$.
- Step 2: Since $m = 8$ is even, Alicia sets $n = \frac{1}{2}m = 4$.
- Step 3: Alicia writes down $m + n + 1 = 13$ as the third term.
- Step 4: Alicia sets $m = 13$.
- Step 2: Since $m = 13$ is odd, Alicia sets $n = m + 1 = 14$.
- Step 3: Alicia writes down $m + n + 1 = 28$ as the fourth term.
- Step 4: Alicia sets $m = 28$.
- Step 2: Since $m = 28$ is even, Alicia sets $n = \frac{1}{2}m = 14$.
- Step 3: Alicia writes down $m + n + 1 = 43$ as the fifth term.
- Step 5: Since Alicia has written down five terms, she stops.

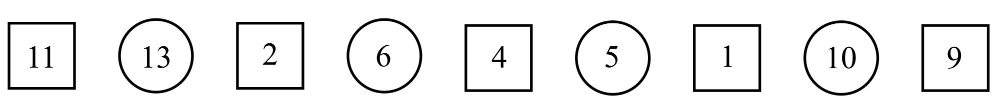
Therefore, the fifth term is 43.

ANSWER: 43

22. From the given information, if a and b are in two consecutive squares, then $a + b$ goes in the circle between them.
Since all of the numbers that we can use are positive, then $a + b$ is larger than both a and b.
This means that the largest integer in the list, which is 13, cannot be either x or y (and in fact cannot be placed in any square). This is because the number in the circle next to it must be smaller than 13 (which is the largest number in the list) and so cannot be the sum of 13 and another positive number from the list.
Thus, for $x + y$ to be as large as possible, we would have x and y equal to 10 and 11 in some order. But here we have the same problem: there is only one larger number from the list (namely 13) that can go in the circles next to 10 and 11, and so we could not fill in the circle next to both 10 and 11.
Therefore, the next largest possible value for $x + y$ is when $x = 9$ and $y = 11$. (We could also swap x and y.)
Here, we could have $13 = 11 + 2$ and $10 = 9 + 1$, giving the following partial list:



The remaining integers (4, 5 and 6) can be put in the shapes in the following way that satisfies the requirements.



This tells us that the largest possible value of $x + y$ is 20.

ANSWER: 20

23. Suppose that Dewa's four numbers are w, x, y, z.
The averages of the four possible groups of three of these are

$$\frac{w+x+y}{3}, \frac{w+x+z}{3}, \frac{w+y+z}{3}, \frac{x+y+z}{3}$$

These averages are equal to 32, 39, 40, 44, in some order.
The sums of the groups of three are equal to 3 times the averages, so are 96, 117, 120, 132, in some order.
In other words, $w+x+y$, $w+x+z$, $w+y+z$, $x+y+z$ are equal to 96, 117, 120, 132 in some order.
Therefore,

$$(w+x+y)+(w+x+z)+(w+y+z)+(x+y+z) = 96+117+120+132$$

and so

$$3w+3x+3y+3z = 465$$

which gives

$$w+x+y+z = 155$$

Since the sum of the four numbers is 155 and the sums of groups of 3 are 96, 117, 120, 132, then the four numbers are

$$155-96=59 \qquad 155-117=38 \qquad 155-120=35 \qquad 155-132=23$$

and so the largest number is 59.

ANSWER: 59

24. Triangular-based pyramid $APQR$ can be thought of as having triangular base $\triangle APQ$ and height AR.
Since this pyramid is built at a vertex of the cube, then $\triangle APQ$ is right-angled at A and AR is perpendicular to the base.
The area of $\triangle APQ$ is $\frac{1}{2} \times AP \times AQ = \frac{1}{2}x(x+1)$. The height of the pyramid is $\frac{x+1}{2x}$.
Thus, the volume of the pyramid is $\frac{1}{3} \times \frac{1}{2}x(x+1) \times \frac{x+1}{2x}$ which equals $\frac{(x+1)^2}{12}$.
Since the cube has edge length 100, its volume is 100^3 or 1 000 000.
Now, 1% of 1 000 000 is $\frac{1}{100}$ of 1 000 000 or 10 000.
Thus, 0.01% of 1 000 000 is $\frac{1}{100}$ of 10 000 or 100.
This tells us that 0.04% of 1 000 000 is 400, and 0.08% of 1 000 000 is 800.
We want to determine the number of integers x for which $\frac{(x+1)^2}{12}$ is between 400 and 800.
This is equivalent to determining the number of integers x for which $(x+1)^2$ is between $12 \times 400 = 4800$ and $12 \times 800 = 9600$.
Since $\sqrt{4800} \approx 69.28$ and $\sqrt{9600} \approx 97.98$, then the perfect squares between 4800 and 9600 are $70^2, 71^2, 72^2, \ldots, 96^2, 97^2$.
These are the possible values for $(x+1)^2$ and so the possible values for x are $69, 70, 71, \ldots, 95, 96$.
There are $96 - 69 + 1 = 28$ values for x.

ANSWER: 28

25. Since the median of the list a, b, c, d, e is 2023 and $a \leq b \leq c \leq d \leq e$, then $c = 2023$.
Since 2023 appears more than once in the list, then it appears 5, 4, 3, or 2 times.

Case 1: 2023 appears 5 times
Here, the list is 2023, 2023, 2023, 2023, 2023.
There is 1 such list.

Case 2: 2023 appears 4 times
Here, the list would be 2023, 2023, 2023, 2023, x where x is either less than or greater than 2023.
Since the mean of the list is 2023, the sum of the numbers in the list is 5×2023, which means that $x = 5 \times 2023 - 4 \times 2023 = 2023$, which is a contradiction.
There are 0 lists in this case.

Case 3: 2023 appears 3 times
Here, the list is a, b, 2023, 2023, 2023 (with $a < b < 2023$) or a, 2023, 2023, 2023, e (with $a < 2023 < e$), or 2023, 2023, 2023, d, e (with $2023 < d < e$).
In the first case, the mean of the list is less than 2023, since the sum of the numbers will be less than 5×2023.
In the third case, the mean of the list is greater than 2023, since the sum of the numbers will be greater than 5×2023.
So we need to consider the list a, 2023, 2023, 2023, e with $a < 2023 < e$.
Since the mean of this list is 2023, then the sum of the five numbers is 5×2023, which means that $a + e = 2 \times 2023$.
Since a is a positive integer, then $1 \leq a \leq 2022$. For each such value of a, there is a corresponding value of e equal to $4046 - a$, which is indeed greater than 2023.
Since there are 2022 choices for a, there are 2022 lists in this case.

Case 4A: 2023 appears 2 times; $c = d = 2023$
(We note that if 2023 appears 2 times, then since $c = 2023$ and $a \leq b \leq c \leq d \leq e$, we either have $c = d = 2023$ or $b = c = 2023$.)
Here, the list is a, b, 2023, 2023, e with $1 \leq a < b < 2023 < e$.
This list has median 2023 and no other integer appears more than once. Thus, it still needs to satisfy the condition about the mean.
For this to be the case, the sum of its numbers equals 5×2023, which means that $a + b + e = 3 \times 2023 = 6069$.
Every pair of values for a and b with $1 \leq a < b < 2023$ will give such a list by defining $e = 6069 - a - b$. (We note that since $a < b < 2023$ we will indeed have $e > 2023$.)
If $a = 1$, there are 2021 possible values for b, namely $2 \leq b \leq 2022$.
If $a = 2$, there are 2020 possible values for b, namely $3 \leq b \leq 2022$.
Each time we increase a by 1, there will be 1 fewer possible value for b, until $a = 2021$ and $b = 2022$ (only one value).
Therefore, the number of pairs of values for a and b in this case is

$$2021 + 2020 + \cdots + 2 + 1 = \tfrac{1}{2} \times 2021 \times 2022 = 2021 \times 1011$$

This is also the number of lists in this case.

Case 4B: 2023 appears 2 times; $b = c = 2023$
Here, the list is a, 2023, 2023, d, e with $1 \leq a < 2023 < d < e$.
This list has median 2023 and no other integer appears more than once. Thus, it still needs to satisfy the condition about the mean.
For this to be the case, the sum of its numbers equals 5×2023, which means that $a + d + e = 3 \times 2023 = 6069$.
If $d = 2024$, then $a + e = 4045$. Since $1 \leq a \leq 2022$ and $2025 \leq e$, we could have $e = 2025$ and $a = 2020$, or $e = 2026$ and $a = 1019$, and so on. There are 2020 such pairs, since once a reaches 1, there are no more possibilities.
If $d = 2025$, then $a + e = 4044$. Since $1 \leq a \leq 2022$ and $2026 \leq e$, we could have $e = 2026$ and $a = 2018$, or $e = 2027$ and $a = 1017$, and so on. There are 2018 such pairs.
As d increases successively by 1, the sum $a + e$ decreases by 1 and the minimum value for e increases by 1, which means that the maximum value for a decreases by 2, which means that the number of pairs of values for a and e decreases by 2. This continues until we reach $d = 3033$ at which point there are 2 pairs for a and e.
Therefore, the number of pairs of values for a and e in this case is

$$2020 + 2018 + 2016 + \cdots + 4 + 2$$

which is equal to

$$2 \times (1 + 2 + \cdots + 1008 + 1009 + 1010)$$

which is in turn equal to $2 \times \frac{1}{2} \times 1010 \times 1011$ which equals1010×1011.

Combining all of the cases, the total number of lists a, b, c, d, e is

$$N = 1 + 2022 + 2021 \times 1011 + 1010 \times 1011 = 1 + 1011 \times (2 + 2021 + 1010) = 1 + 1011 \times 3033$$

and so $N = 3\,066\,364$.
The sum of the digits of N is $3 + 0 + 6 + 6 + 3 + 6 + 4$ or 28.

Answer: 28