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2021 Canadian Team Mathematics Contest
Answer Key for Team Problems

Question	Answer
1	59
2	18
3	198
4	25
5	6
6	22
7	54
8	2
9	75
10	8
11	\$100
12	-7
13	9
14	Ann
15	13
16	17
17	161
18	270°
19	32
20	$\frac{16}{5}$
21	$\frac{1}{\sqrt{2}}$
22	$p+1$
23	49
24	80, 96
25	$\frac{1}{10}$

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Answer Key for Individual Problems

Question	Answer
1	$\frac{11}{24}$
2	30°
3	12
4	1012
5	243
6	204
7	4491
8	11
9	$\frac{3}{2}$
10	83

Answer Key for Relays

Question	Answer
0	9, 108, 36°
1	$\left(7, -4, \frac{9}{2}\right)$
2	$(4, 15, 3)$
3	$(13, 11, 25)$

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2022
Canadian Team Mathematics Contest

April 2022

Solutions

Individual Problems

1. From the bar graph, 11 students chose chocolate ice cream, 5 students chose strawberry ice cream, and 8 students chose vanilla ice cream.
   The total number of students is $11 + 5 + 8 = 24$, so the answer is $\frac{11}{24}$.

   Answer: $\frac{11}{24}$

2. Since $EB = BC = CE$, $\triangle BCE$ is equilateral so $\angle BEC = 60°$.
   Since DF is parallel to EB, $\angle FDC = \angle BEC = 60°$.
   Since AB is parallel to DC and AD is perpendicular to AB, AD is also perpendicular to DC, so $\angle ADC = 90°$.
   Therefore, $\angle FDA = \angle ADC - \angle FDC = 90° - 60° = 30°$.

   Answer: $30°$

3. Let x the real number satisfying $BC = 6x$.
   From $AB : BC = 1 : 2$ we get that $AB = 3x$.
   From $BC : CD = 6 : 5$ we get that $CD = 5x$.
   This means $AD = AB + BC + CD = 3x + 6x + 5x = 14x$.
   We are given that $AD = 56$, so $56 = 14x$ or $x = 4$.
   Therefore, $AB = 3x = 3(4) = 12$.

   Answer: 12

4. The terms being added and subtracted are the integers $2(1)$, $2(2)$, $2(3)$, and so on up to $2(1011)$.
   This means there are 1011 terms in total, which is an odd number of terms.
   We will now insert parentheses to group the terms in pairs starting with the first two terms, then the second two terms, and so on.
   Since the total number of terms is odd, 2022 will not be grouped with another term:

   $$S = (2 - 4) + (6 - 8) + (10 - 12) + \cdots + (2014 - 2016) + (2018 - 2020) + 2022$$

   Each of the parenthetical expressions is equal to -2, and the number of parenthetical expressions is $\frac{1011 - 1}{2} = 505$.
   Therefore,

   $$S = 505(-2) + 2022 = -1010 + 2022 = 1012$$

   Answer: 1012

5. Suppose $12n = k^2$ for some integer k. Then k^2 is even and so k must be even, which means $\frac{k}{2}$ is an integer.
   Dividing both sides of $12n = k^2$ by 4 gives $3n = \left(\frac{k}{2}\right)^2$, and since $\frac{k}{2}$ is an integer, this means $3n$ is a perfect square.
   We are given that $200 < n < 250$, which implies $600 < 3n < 750$.
   The perfect squares between 600 and 750 are 625, 676, and 729, among which 729 is the only multiple of 3, so $3n = 729$ or $n = 243$.
   This value of n satisfies $200 < n < 250$, and $12n = 2916 = 54^2$.
   We have shown that $n = 243$ is the only n that satisfies the conditions, so it must be the largest.

   Answer: 243

6. Substituting $B = 3C$ into $D = 2B - C$ gives $D = 2(3C) - C = 5C$.
Substituting $D = 5C$ and $B = 3C$ into $A = B + D$ gives $A = 3C + 5C = 8C$.
Since $A = 8C$ is a two-digit integer, we must have $10 \leq 8C \leq 99$, so $\dfrac{10}{8} \leq C \leq \dfrac{99}{8}$.
Since C is an integer, this implies $2 \leq C \leq 12$.
We also know that C is a two-digit integer, so $10 \leq C$, which means that $C = 10$, $C = 11$, or $C = 12$.
Using the equations from earlier, we have $A + B + C + D = 8C + 3C + C + 5C = 17C$, so the larger C is, the larger $A + B + C + D$ is.
With $C = 12$, we get $A = 8(12) = 96$, $B = 3(12) = 36$, and $D = 5(12) = 60$, which are all two-digit integers.
Therefore, the conditions are satisfied when $C = 12$, its largest possible value, which means $C = 12$ corresponds to the largest possible value of $A + B + C + D$.
Therefore, the answer is $A + B + C + D = 17C = 17(12) = 204$.

ANSWER: 204

7. We can rewrite the given expression as follows:

$$\begin{aligned} 3 \times 10^{500} - 2022 \times 10^{497} - 2022 &= 3000 \times 10^{497} - 2022 \times 10^{497} - 2022 \\ &= (3000 - 2022) \times 10^{497} - 2022 \\ &= 978 \times 10^{497} - 2022 \\ &= \left(978 \times 10^{497} - 1\right) - 2021 \end{aligned}$$

The integer $978 \times 10^{497} - 1$ is the 500-digit integer with leading digits 977 followed by 497 9's.
Thus, the digits of the given integer, from left to right, are 9, 7, and 7 followed by $497 - 4 = 493$ 9's, and the final four digits are $9 - 2 = 7$, $9 - 0 = 9$, $9 - 2 = 7$, and $9 - 1 = 8$.
Therefore, the sum of the digits of the integer is

$$9 + 7 + 7 + (493 \times 9) + 7 + 9 + 7 + 8 = 4491$$

ANSWER: 4491

8. By rearranging, we get

$$\begin{aligned} \frac{2n^3 + 3n^2 + an + b}{n^2 + 1} &= \frac{2n^3 + 2n + 3n^2 + 3 + (a-2)n + (b-3)}{n^2 + 1} \\ &= \frac{2n(n^2 + 1) + 3(n^2 + 1) + (a-2)n + (b-3)}{n^2 + 1} \\ &= 2n + 3 + \frac{(a-2)n + (b-3)}{n^2 + 1} \end{aligned}$$

The quantity $2n + 3$ is an integer if n is an integer, so the given expression in n is an integer exactly when $\dfrac{(a-2)n + (b-3)}{n^2 + 1}$ is an integer.
Suppose that $a - 2 \neq 0$ and consider the functions $f(x) = x^2 + 1$, $g(x) = (a-2)x + b - 3$, and $h(x) = f(x) - g(x) = x^2 - (a-2)x + 4 - b$.
The graph of $h(x)$ is a parabola with a positive coefficient of x^2, which means that there are finitely many integers n with the property that $h(n) \leq 0$ (there could be no such integers).
Since $h(n) = f(n) - g(n)$, we have shown that there are only finitely many integers n for which $g(n) < f(n)$ is false.
The graph of $g(x)$ is a line that is neither vertical nor horizontal since $a - 2 \neq 0$, so there must

be infinitely many integers for which $g(n) > 0$.
Since infinitely many integers satisfy $0 < g(n)$ and only finitely many integers fail $g(n) < f(n)$, there must exist an integer n with the property that $0 < g(n) < f(n)$.
This means there exists an integer n for which

$$0 < \frac{g(n)}{f(n)} < 1$$

but this means

$$\frac{g(n)}{f(n)} = \frac{(a-2)n + b - 3}{n^2 + 1}$$

is not an integer. Therefore, we have shown that if $a - 2 \neq 0$, then there is an integer n for which the expression in the problem is not an integer, so we conclude that $a - 2 = 0$ or $a - 2$.
We now have that the expression in the question is equivalent to

$$2n + 3 + \frac{b-3}{n^2+1}$$

which is an integer exactly when $\dfrac{b-3}{n^2+1}$ is an integer.
If $b - 3$ is positive, then n can be chosen so that $n^2 + 1$ is larger than $b - 3$, which would mean that the positive expression $\dfrac{b-3}{n^2+1}$ is not an integer.
Similarly, if $b - 3$ is negative, then there are integers n for which $\dfrac{b-3}{n^2+1}$ is not an integer.
This shows that if $b - 3 \neq 0$, then there are integers n for which the expression in the question is not an integer.
We can conclude that $b - 3 = 0$, or $b = 3$.
Therefore, we have that $a = 2$ and $b = 3$, so the expression given in the problem is equivalent to

$$2n + 3 + \frac{(2-2)n + (3-3)}{n^2+1} = 2n + 3$$

Thus, the answer to the question is $2(4) + 3 = 11$.

Answer: 11

9. We will use the fact that if two circles are externally tangent, then the line connecting their centres passes through the point of tangency.
The side lengths of $\triangle ABC$ are $AB = (\sqrt{3}-1)+(3-\sqrt{3}) = 2$, $AC = (\sqrt{3}-1)+(1+\sqrt{3}) = 2\sqrt{3}$, and $BC = (3-\sqrt{3}) + (1+\sqrt{3}) = 4$.
Notice that

$$\begin{aligned} AB^2 + AC^2 &= 2^2 + (2\sqrt{3})^2 \\ &= 4 + 12 \\ &= 16 \\ &= 4^2 \\ &= BC^2 \end{aligned}$$

and so the sides of $\triangle ABC$ satisfy $BC^2 = AB^2 + AC^2$. By the *converse* of the Pythagorean theorem, $\triangle ABC$ has $\angle CAB = 90°$.
Therefore, the area of $\triangle ABC$ is $\dfrac{1}{2} \times AB \times AC = \dfrac{1}{2} \times 2 \times 2\sqrt{3} = 2\sqrt{3}$.

Since $\angle CAB = 90°$, the unshaded sector that lies inside the triangle has area $\dfrac{90°}{360°} = \dfrac{1}{4}$ of the area of the circle centred at A.
The area of the circle centred at A is $\pi(\sqrt{3}-1)^2 = \pi(4-2\sqrt{3})$.
The area of the shaded region is therefore

$$2\sqrt{3} - \frac{\pi(4-2\sqrt{3})}{4} = 2\sqrt{3} - \pi + \frac{1}{2}\pi\sqrt{3}$$

which means $a = 2$, $b = -1$, and $c = \dfrac{1}{2}$, so $a + b + c = \dfrac{3}{2}$.

ANSWER: $\frac{3}{2}$

10. We will count the integers with the desired property by considering four cases, one for each possible leading digit.
The leading digit is 1: In this situation, we are looking for integers of the form $1000 + n$ where n is an integer from 1 to 999 inclusive so that $1000 + n$ is a multiple of n.
For $1000 + n$ to be a multiple of n, there must be some integer k so that $kn = 1000 + n$, which can be rearranged to get $n(k-1) = 1000$.
Thus, $1000 + n$ will have the desired property exactly when n is a positive factor of 1000 that is less than 1000.
The positive factors of 1000 that are less than 1000 are

$$1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 125, 200, 250, 500$$

There are 15 integers in the list above, so we get 15 integers in this case.

The leading digit is 2: Similar to case 1, we need to find all integers n for which $1 \leq n \leq 999$ and $2000 + n$ is a multiple of n.
The latter condition means there is an integer k with the property that $2000 + n = kn$ which can be rearranged to get $2000 = n(k-1)$.
The positive factors of 2000 that are less than 1000 are

$$1, 2, 4, 5, 8, 10, 16, 20, 25, 40, 50, 80, 100, 125, 200, 250, 400, 500$$

of which there are 18. Therefore, there are 18 integers in this case.

The leading digit is 3: As in cases 1 and 2, we need to find all integers n for which $1 \leq n \leq 999$ and $3000 + n$ is a multiple of n.
Again, this means we need to count the positive factors of 3000 that are less than 1000.
If m is a factor of 3000, then it is either a multiple of 3 or it is not.
If it is not a multiple of 3, then it is one of the 15 factors of 1000 from case 1.
Otherwise, it is 3 times one of the factors of 1000 from case 1.
Among the factors from case 1, all except 500 remain less than 1000 when multiplied by 3.
Therefore, there are $15 + 14 = 29$ integers in this case.

The leading digit is 4: In this case, we need to count the positive factors of 4000 that are smaller than 1000.
The prime factorization of 4000 is $2^5 5^3$.
Each factor of 4000 is either a power of 2, 5 times a power of 2, 25 times a power of 2, or 125 times a power of 2.

Each of $1 = 2^0$, $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, and $32 = 2^5$ is less than 1000, so 6 of the factors are powers of 2.
Multiplying each of the powers of 2 in the previous line by 5 gives 5, 10, 20, 40, 80, and 160, each of which is less than 1000.
Multiplying 5 by each of the factors in the previous line gives 25, 50, 100, 200, 400, and 800, each of which is less than 1000.
Multiplying 5 by each of the factors in the previous line gives 125, 250, 500, 1000, 2000, 4000, three of which are less than 1000.
This gives a total of $6 + 6 + 6 + 3 = 21$ factors of 4000 that are less than 1000, so we get 21 integers in this case.

Thus, the number of integers with the desired property is $15 + 18 + 29 + 21 = 83$.

ANSWER: 83

Team Problems

1. Substituting $x = 40$ into $x = 2z$ gives $40 = 2z$ so $z = 20$.
   Substituting $z = 20$ into $y = 3z - 1$ gives $y = 3(20) - 1 = 59$.

   ANSWER: 59

2. The smallest two positive two-digit multiples of 6 are 12 and 18.
   Since 12 is a multiple of 4 and 18 is not, the answer is 18.

   ANSWER: 18

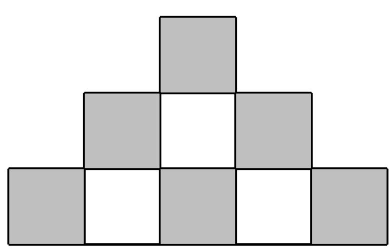
3. Since the first digit of a four-digit integer cannot be 0, there are three integers with the given property: 2022, 2202, and 2220.
   The largest is 2220 and the smallest is 2022, so the answer is $2220 - 2022 = 198$.

   ANSWER: 198

4. Let Y denote the number of people who answered "yes" and let N denote the number of people who answered "no".
   The number of people is n, so $n = Y + N$. As well, since 76% of respondents said "yes", we have that $\frac{76}{100} = \frac{Y}{n} = \frac{Y}{Y+N}$.
   The equation $\frac{76}{100} = \frac{Y}{Y+N}$ implies $76Y + 76N = 100Y$ which can be rearranged to $76N = 24Y$.
   Dividing both sides by 4 gives $19N = 6Y$.
   Since Y and N are numbers of people, they are non-negative integers. Furthermore, since a non-zero percentage of people responded with each of "yes" and "no", they must be positive integers.
   Therefore, the equation $19N = 6Y$ implies that Y is a positive multiple of 19 since 6 and 19 do not have any positive factors in common other than 1.
   The smallest that Y can be is 19, and if $Y = 19$, then $N = \frac{(6)(19)}{19} = 6$.
   If $Y = 19$ and $N = 6$, then $n = 25$, which means $\frac{Y}{n} = \frac{19}{25} = 0.76$ and $\frac{N}{n} = \frac{6}{25} = 0.24$.
   Therefore, $Y = 19$ and $N = 6$ satisfy the conditions, and we have already argued that Y cannot be any smaller.
   If Y is greater than 19, then n is greater than $2(19) = 38$ since Y must be a positive multiple of 19. Therefore, the answer is 25.

   ANSWER: 25

5. In the diagram below, there are 6 squares shaded in such a way that no two shaded squares share an edge, which shows that the answer is at least 6.



We will now argue that it is impossible to shade more than 6 square in such a way that no two shaded squares share an edge, which will show that the answer to the question is 6.
If four or five squares in the bottom row are shaded, then at least two shaded squares will share an edge.
Therefore, at most 3 squares can be shaded in the bottom row.
If all three squares in the middle row are shaded, then there will be shaded squares that share an edge.
Therefore, at most 2 squares can be shaded in the middle row.
There is only one square in the top row, so at most 1 square can be shaded in the top row.
Thus, the number of squares that can be shaded so that no two shaded squares share an edge is at most $3 + 2 + 1 = 6$.

ANSWER: 6

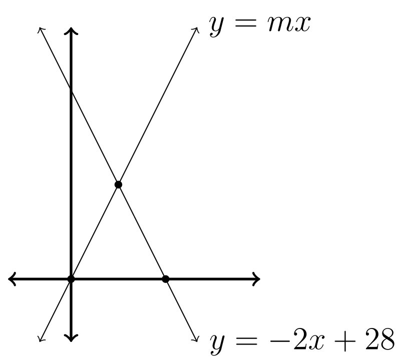
6. Multiplying both sides of the equation by $12x$ gives $12 + 6 + 4 = x$, so $x = 22$.

ANSWER: 22

7. An isosceles triangle must have at least two sides of equal length.
If an isosceles triangle has at least one side of length 10 and at least one side of length 22, then its three side lengths are either 10, 10, and 22 or 10, 22, and 22.
In any triangle, the sum of the lengths of any two sides must be larger than the length of the other side.
Since $10 + 10 < 22$, a triangle cannot have sides of length 10, 10, and 22.
Therefore, the triangle's sides have lengths 10, 22, and 22, so the answer is $10 + 22 + 22 = 54$.

ANSWER: 54

8. The area of interest is a triangle with vertices at the origin, the x-intercept of $y = -2x + 28$, and the point of intersection of the lines with equations $y = mx$ and $y = -2x + 28$.



Setting $y = 0$ in $y = -2x + 28$ gives $0 = -2x + 28$ which implies $x = 14$.
Therefore, the base of the triangle has length $14 - 0 = 14$.
Setting $mx = -2x+28$ leads to $x = \dfrac{28}{m+2}$, and substituting this into $y = mx$ gives $y = \dfrac{28m}{m+2}$, which is the height of the triangle.
Using that the area of the triangle is 98, we have that $98 = \dfrac{1}{2} \times 14 \times \dfrac{28m}{m+2}$ or $98 = \dfrac{196m}{m+2}$.
Dividing both sides of this equation by 98 gives $1 = \dfrac{2m}{m+2}$, which can be rearranged to get $m + 2 = 2m$, which implies that $m = 2$.

ANSWER: 2

9. The side length of a cube with volume $a^3\,\text{cm}^3$ is $a\,\text{cm}$.
Since $10^3 = 1000$ and $4^3 = 64$, the side length of the original cube is $10\,\text{cm}$ and the side length of the smaller cube is $4\,\text{cm}$.
The new figure has nine faces: three $10\,\text{cm}\times 10\,\text{cm}$ squares, three $4\,\text{cm}\times 4\,\text{cm}$ squares, and three faces obtained by removing a $4\,\text{cm}\times 4\,\text{cm}$ square from the corner of a $10\,\text{cm}\times 10\,\text{cm}$ square.
Thus, the surface area of the new figure in cm^2 is

$$3(10^2) + 3(4^2) + 3(10^2 - 4^2) = 600$$

[Note that this is exactly the surface area of the original cube. Can you see a simple reason why this is true?]
The shaded region is a $4\,\text{cm}\times 4\,\text{cm}$ square, so the ratio we seek is $4^2 : 600$ or $2 : 75$.
Therefore, $x = 75$.

ANSWER: 75

10. The expression $\dfrac{8(n-1)}{(n-1)(n-2)}$ is undefined when $n = 1$. For every integer $n \neq 1$, the expression is equal to $\dfrac{8}{n-2}$.
Thus, if n has the property that $\dfrac{8(n-1)}{(n-1)(n-2)}$ is also an integer, then n must be an integer different from 1 with the property that $n-2$ is a factor of 8.
The factors of 8 are

$$-8, -4, -2, -1, 1, 2, 4, 8$$

The following table has these values of $n-2$ in the first column, the corresponding values of n in the second column, and the corresponding values of $\dfrac{8(n-1)}{(n-1)(n-2)}$ in the third column.

$n-2$	n	$\dfrac{8(n-1)}{(n-1)(n-2)}$
-8	-6	-1
-4	-2	-2
-2	0	-4
-1	1	undefined
1	3	8
2	4	4
4	6	2
8	10	1

Other than when $n = 1$, we get an integer value of $\dfrac{8(n-1)}{(n-1)(n-2)}$.
The sum of the integer values of $\dfrac{8(n-1)}{(n-1)(n-2)}$ is $-1 - 2 - 4 + 8 + 4 + 2 + 1 = 8$.

ANSWER: 8

11. Let a be the amount in dollars initially invested in Account A, let b be the amount in dollars initially invested in Account B, and let c be the amount in dollars initially invested in Account C.
The given information leads to the equations $a + b + c = 425$, $0.05a = 0.08b$, and $0.08b = 0.1c$.
The second and third equations can be solved for a and b, respectively, to get $a = \dfrac{8}{5}b$ and

$b = \frac{5}{4}c$.

Substituting $b = \frac{5}{4}c$ into the equation $a = \frac{8}{5}b$, we get $a = 2c$.

Substituting $a = 2c$ and $b = \frac{5}{4}c$ into the equation $a + b + c = 425$ gives

$$425 = a + b + c = 2c + \frac{5}{4}c + c = \frac{17}{4}c$$

Therefore, the amount in dollars initially invested in Account C is $c = \frac{4}{17} \times 425 = 100$.

ANSWER: \$100

12. *Solution 1*

The original line contains the points $(0, 6)$ and $(1, 9)$.
Translating these points 3 units up gives the points $(0, 9)$ and $(1, 12)$.
Translating these points 4 units to the left gives $(-4, 9)$ and $(-3, 12)$.
Reflecting these points in the line $y = x$ gives $(9, -4)$ and $(12, -3)$.
The slope of the line through these two points is $\frac{-3-(-4)}{12-9} = \frac{1}{3}$.
Substituting $x = 9$ and $y = -4$ into $y = \frac{1}{3}x + b$ gives $-4 = 3 + b$ or $b = -7$.
Therefore, the equation of the resulting line is $y = \frac{1}{3}x - 7$, so the answer is -7.

Solution 2

The equation of the line obtained by translating up by 3 units is $y = 3x + 6 + 3$ or $y = 3x + 9$.
The equation of the line obtained by translating to the left by 4 units is $y = 3(x + 4) + 9$ or $y = 3x + 21$.
To find the equation of the line obtained by reflecting the line in $y = x$, we interchange x and y to get $x = 3y + 21$, then solve for y to get $y = \frac{1}{3}x - 7$. Therefore, the answer is -7.

ANSWER: -7

13. The sum of the integers in the list is

$$(m+1) + 2(m+2) + 3(m+3) + 4(m+4) + 5(m+5) = 15m + 1 + 4 + 9 + 16 + 25 = 15m + 55$$

The number of integers in the list is

$$(m+1) + (m+2) + (m+3) + (m+4) + (m+5) = 5m + 15$$

The average of the integers in the list is $\frac{19}{6}$, so this means

$$\frac{19}{6} = \frac{15m+55}{5m+15} = \frac{3m+11}{m+3}$$

The above equation is equivalent to $19(m+3) = 6(3m+11)$, or $19m + 57 = 18m + 66$, which can be solved for m to get $m = 9$.

ANSWER: 9

14. Suppose Devi was given the bill for Bohan's meal. From the second bullet point, this would mean that Bohan and Devi were given each other's bills.
This means that Ann's, Che's, and Eden's bills were distributed among each-other in some way.
Che was given Ann's bill by the fourth bullet point, so if Ann was given Che's bill, then Eden would have been given her own bill.
Since nobody was given their own bill, Ann must have been given Eden's bill, so the only bill remaining for Eden to have been given is Che's.
Thus, assuming that Devi was given Bohan's bill, we have deduced that Ann was given Eden's bill, Eden was given Che's bill, and Che was given Ann's bill.
By the first bullet point, Che and Eden ordered meals with the same cost.
Since Eden was given Che's bill, the third bullet point is violated.

This means that Devi was not given Bohan's bill.
Because Che was given Ann's bill, Devi was not given Ann's bill.
As well, Devi could not have been given Che's bill because this would imply, by the second bullet point, that Che was given Devi's bill, which would violate the fourth bullet point.

By eliminating all other possibilities, we have deduced that Devi's and Eden's bills were interchanged.
This means that the remaining three bills, belonging to Ann, Bohan, and Che, were distributed among the same three people.
Similar to the reasoning from before, since nobody was given their own bill and Che was given Ann's bill, the only possibility is that Bohan was given Che's bill and Ann was given Bohan's bill.

ANSWER: Ann

15. *Solution 1*

The line segment PQ has slope $\dfrac{-3}{9} = -\dfrac{1}{3}$.
Since $\angle PQR = 90^\circ$, the slope of QR is the negative reciprocal of $-\dfrac{1}{3}$, which is 3.
The line segment QR has slope $\dfrac{3-(-3)}{x-11} = \dfrac{6}{x-11}$, so $\dfrac{6}{x-11} = 3$.
This implies $6 = 3x - 33$, so $3x = 39$ or $x = 13$.

Solution 2

In any right-angled triangle, the three vertices are equidistant from the midpoint of the hypotenuse.
The right angle is at Q, so this means that the hypotenuse of $\triangle PQR$ is PR.
The coodinates of P are $(2, 0)$ and the coordinates of R are $(x, 3)$, so the coordinates of the midpoint of PR is at $\left(\dfrac{2+x}{2}, \dfrac{3}{2}\right)$.
The length of PR is

$$\sqrt{(x-2)^2 + 3^2} = \sqrt{x^2 - 4x + 13}$$

The distance from Q to the midpoint of PR is half of $\sqrt{x^2 - 4x + 13}$, which leads to the equations

$$\begin{aligned}
\frac{1}{2}\sqrt{x^2-4x+13} &= \sqrt{\left(11-\frac{2+x}{2}\right)^2+\left(-3-\frac{3}{2}\right)^2} \\
\sqrt{x^2-4x+13} &= 2\sqrt{\left(\frac{20-x}{2}\right)^2+\left(-\frac{9}{2}\right)^2} \\
x^2-4x+13 &= 2^2\left(\frac{20-x}{2}\right)^2+2^2\left(-\frac{9}{2}\right)^2 \\
x^2-4x+13 &= (20-x)^2+(-9)^2 \\
x^2-4x+13 &= x^2-40x+400+81 \\
36x &= 468 \\
x &= 13
\end{aligned}$$

It can be checked that if the coordinates of R are $(13,3)$, that the side lengths of $\triangle PQR$ are $\sqrt{40}$, $\sqrt{90}$, and $\sqrt{130}$, which are indeed the lengths of the sides of a right-angled triangle.

ANSWER: 13

16. Using that 3 is a solution to $x^2-7x+k=0$ we have that $3^2-7(3)+k=0$ or $k=21-9=12$. This means x^2-7x+k is $x^2-7x+12=(x-3)(x-4)$, the roots of which are 3 and 4, which means $a=4$.
The polynomial $x^2-8x+k+1$ factors as $(x-b)(x-c)$, so upon expanding the latter expression, we have $bc=k+1=12+1=13$.
Therefore, $a+bc=4+13=17$.

ANSWER: 17

17. For positive x,

$$\begin{aligned}
g(f(x)) &= \log_3(9f(x)) \\
&= \log_3(9\times 9^x) \\
&= \log_3(9^{x+1}) \\
&= \log_3((3^2)^{x+1}) \\
&= \log_3(3^{2x+2}) \\
&= (2x+2)\log_3(3) \\
&= 2x+2
\end{aligned}$$

A similar calculation shows that

$$\begin{aligned}
f(g(2)) &= 9^{g(2)} \\
&= 9^{\log_3(18)} \\
&= 3^{2\log_3(18)} \\
&= \left(3^{\log_3(18)}\right)^2 \\
&= 18^2 \\
&= 324
\end{aligned}$$

Therefore, $2x+2=324$ so $x=\dfrac{324-2}{2}=161$.

Answer: 161

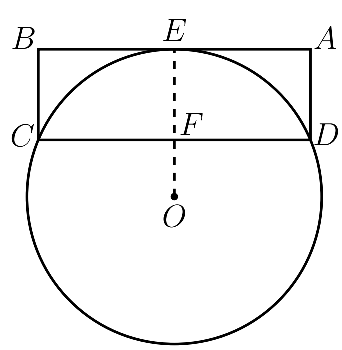
18. If $2022^x = 1$, then $x = 0$. This means we have $2\sin^2\theta - 3\sin\theta + 1 = 0$.
The expression on the left can be factored to get $(\sin\theta - 1)(2\sin\theta - 1) = 0$.
Therefore, the given equation is true exactly when $\sin\theta = 1$ or $\sin\theta = \frac{1}{2}$.
Since $0^\circ < \theta < 360^\circ$, the only value of θ with $\sin\theta = 1$ is $\theta = 90^\circ$.
The only values of θ with $\sin\theta = \frac{1}{2}$ are $\theta = 30^\circ$ and $\theta = 150^\circ$.
Therefore, the answer is $90^\circ + 30^\circ + 150^\circ = 270^\circ$.

Answer: 270°

19. The distance from point A to each of points B, D, and I is 1.
Since $ABCD$ is a square of side length 1, the Pythagorean theorem implies that the distance from A to C is $\sqrt{1^2 + 1^2} = \sqrt{2}$.
A similar calculation shows that the distance from A to K is $\sqrt{2}$ and the distance from A to O is $\sqrt{2}$.
To compute the distance from A to M, observe that $\triangle AOM$ is right-angled at O with $OM = 1$, so the distance from A to M is $\sqrt{AO^2 + OM^2} = \sqrt{2+1} = \sqrt{3}$.
The distance from A to J is 2 which is greater than $\sqrt{2}$.
Notice that J is the closest point to A among the points J, L, N, P, E, F, G, and H, so none of these points can be a distance of $\sqrt{2}$ from A.
Therefore, the points that are at a distance of $\sqrt{2}$ from A are C, K, and O, for a total of 3.
By symmetry, there will be a total of 3 points at a distance of $\sqrt{2}$ from each of B, C, D, E, F, G, and H.
Next consider point K. By reasoning similar to that which is above, each of E, F, G, and H has a distance of at least $2 > \sqrt{2}$ from K.
Applications of the Pythagorean theorem can be used to show that the points that are $\sqrt{2}$ away from K are A, C, O, N, and J, for a total of 5 points.
By symmetry, there are a total of 5 points at a distance of $\sqrt{2}$ from each of I, J, L, M, N, O, and P.
We have counted 3 points at a distance of $\sqrt{2}$ from each of the 8 points A through H, and 5 points at a distance of $\sqrt{2}$ from each of the 8 points from I through P.
This is a total of $8 \times 3 + 8 \times 5 = 64$.
The lengths of the line segments were counted once for each end point, which means the total of 64 counts every distance of $\sqrt{2}$ twice.
Therefore, the answer is $\frac{64}{2} = 32$.

Answer: 32

20. Label the centre of the circle by O, the point of tangency of AB with the circle by E, and the point of intersection of OE with CD by F:



Let $EF = x$. Then $AD = x$, so $AB = 4x$, which means $CD = 4x$ because $ABCD$ is a rectangle.
Using circle properties, OE is perpendicular to AB, and since AB is parallel to CD, we also have that OE, and hence, OF, is perpendicular to CD.
We now have that $\triangle OFD$ is right-angled at F, and $\triangle OFC$ is also right-angled at F.
Since OC and OD are radii of the same circle, they must be equal.
Triangles OFD and OFC also share side OF, so they are congruent by hypotenuse-side similarity.
Therefore, F is the midpoint of CD, so $DF = 2x$.
It is given that the radius of the circle is $\sqrt{5}$, which implies that $OF = \sqrt{5} - x$ and $OD = \sqrt{5}$.
By the Pythagorean theorem, $OF^2 + DF^2 = OD^2$, and after substituting the lengths computed earlier, this leads to the following equivalent equations:

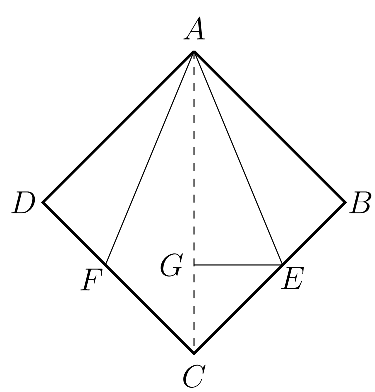
$$\begin{aligned}(\sqrt{5} - x)^2 + (2x)^2 &= \sqrt{5}^2 \\ 5 - 2\sqrt{5}x + x^2 + 4x^2 &= 5 \\ 5x^2 - 2\sqrt{5}x &= 0 \\ x(5x - 2\sqrt{5}) &= 0\end{aligned}$$

this implies that either $x = 0$ or $5x = 2\sqrt{5}$.
We cannot have $x = 0$ since x is the side-length of a rectangle. Therefore, $5x = 2\sqrt{5}$ or $x = \dfrac{2}{\sqrt{5}}$.
The area of $ABCD$ is $x(4x) = \dfrac{2}{\sqrt{5}} \times \dfrac{8}{\sqrt{5}} = \dfrac{16}{5}$.

ANSWER: $\frac{16}{5}$

21. *Solution 1*

Suppose $ABCD$ has side length x and let G be on AC so that EG is perpendicular to AC. Set $GE = y$.



When the paper is folded, B lands on AC and E does not move.
This means that $\angle BAE = \angle GAE$. As well, $\angle AGE = \angle ABE = 90°$ by construction.
Since $\triangle ABE$ and $\triangle AGE$ have two angles in common, they must have three angles in common.
They also share side AE, so they must be congruent.
The area of $AECF$ is the sum of the areas of $\triangle AEC$ and $\triangle AFC$.
By symmetry, $\triangle AFC$ is congruent to $\triangle AEC$, which means the area of $AECF$ is

$$2\left(\frac{1}{2} \times AC \times GE\right) = (AC)(GE)$$

The length of AC is $\sqrt{2}x$ since it is a diagonal of a square with side length x.
Therefore, the area of $AECF$ is $\sqrt{2}xy$.
The area of the shaded region is twice the area of $\triangle AGE$, and since $\triangle AGE$ is congruent to $\triangle ABE$, we have $AG = AB = x$, so the shaded region has area

$$2\left(\frac{1}{2} \times AG \times GE\right) = (AG)(GE) = xy$$

Therefore, the fraction we seek is $\dfrac{xy}{\sqrt{2}xy} = \dfrac{1}{\sqrt{2}}$.

Solution 2

Let the side length of the square be 1, set $BE = x$, and define G as in Solution 1.
Using the argument from Solution 1, we get that $GE = x$ as well.
Since AC is the diagonal of a square, $\triangle ABC$ is isosceles with $\angle ABC = 90°$, which means $\angle ACE = 45°$.
Since $\angle CGE = 90°$ by construction, $\triangle CGE$ is an isosceles right triangle, so $CE = \sqrt{2}GE$.
We also have that $CE = BC - BE = 1 - x$ and $GE = x$, so this means $1 - x = \sqrt{2}x$.
Solving for x, we get $x = \dfrac{1}{\sqrt{2}+1} = \dfrac{1}{\sqrt{2}+1} \times \dfrac{\sqrt{2}-1}{\sqrt{2}-1} = \sqrt{2}-1$.
Using the argument from Solution 1, $\triangle ABE$, $\triangle AGE$, $\triangle AGF$, and $\triangle ADF$ are all congruent.
This means the area of the shaded region is equal to twice the area of $\triangle ABE$, which is

$$2\left(\frac{1}{2} \times AB \times BE\right) = x = \sqrt{2}-1$$

The area of $AEFC$ is the area of the square minus the combined area of $\triangle ABE$ and $\triangle ADF$.
This means the area of $AEFC$ is 1 minus twice the area of $\triangle ABE$, or $1 - (\sqrt{2}-1) = 2 - \sqrt{2}$.
Therefore, the ratio is

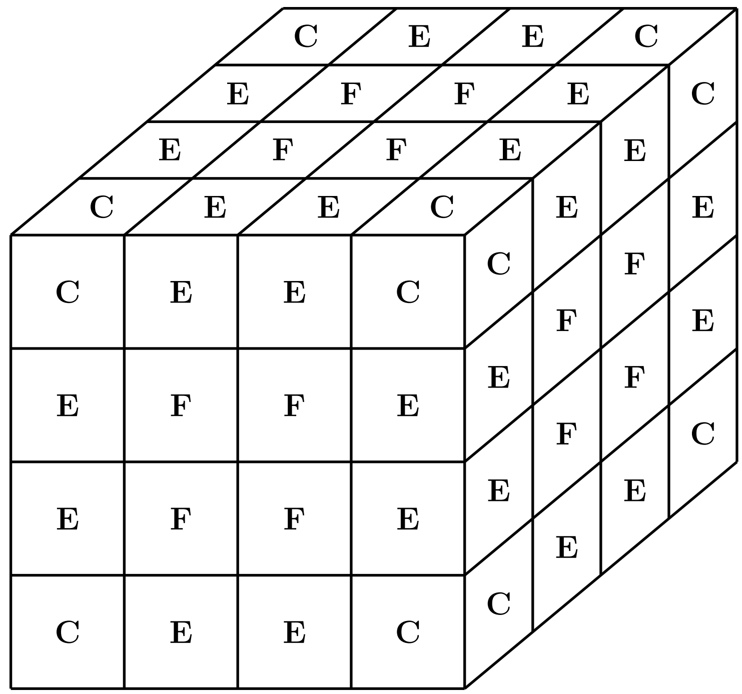
$$\frac{\sqrt{2}-1}{2-\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}(\sqrt{2}-1)} = \frac{1}{\sqrt{2}}$$

ANSWER: $\frac{1}{\sqrt{2}}$

22. Factoring $x^2 + xz - xy - yz$, we have $x(x+z) - y(x+z)$ or $(x-y)(x+z)$.
Therefore, the integers x, y, z, and p satisfy $(x-y)(x+z) = -p$ or $(y-x)(x+z) = p$.
Because p is a prime number and $y-x$ and $x+z$ are integers, it must be that $(y-x, x+z)$ is a factor pair of p.
Observe that $y+z = (y-x)+(x+z)$, so $y+z$ must be equal to the sum of the integers in a factor pair of p.
Since p is a prime number, there are only two ways to express p as the product of two integers, and they are $p = (-1) \times (-p)$ and $p = 1 \times p$.
The sum of the factors in these products are $-p-1 = -(p+1)$ and $p+1$, respectively.
Therefore, $y+z = \pm(p+1)$, so $|y+z| = p+1$.

ANSWER: $p+1$

23. We can categorize the 64 small cubes into four groups: 8 cubes that are completely in the interior of the larger cube (and are completely invisible), $4 \times 6 = 24$ "face cubes" that have exactly one of their faces showing, $2 \times 12 = 24$ "edge cubes" that have exactly two faces showing, and 8 "corner cubes" that have 3 faces showing.
Notice that $8 + 24 + 24 + 8 = 64$, so this indeed accounts for all 64 small cubes.
The diagram below has all visible faces of the smaller cubes labelled so that faces of edge cubes are labelled with an **E**, faces of face cubes are labelled with an **F**, and faces of corner cubes are labelled with a **C**:



Each of the 8 corner cubes has 3 of its faces exposed.
Because of the way the faces are labelled on the small cubes, each of the 8 corner cubes will always contribute a total of $1 + 1 + 2 = 4$ to the sum of the numbers on the outside.
Regardless of how the small cubes are arranged, the corner cubes contribute a total of $8 \times 4 = 32$ to the total of all numbers on the outside of the larger cube.
The edge cubes can each show either a 1 and a 1 or a 1 and a 2, for a total of either 2 or 3.
Thus, each edge cube contributes a total of either 2 or 3 to the total on the outside of the larger cube.
This means the smallest possible total that the edge cubes contribute is $24 \times 2 = 48$, and the largest possible total that they contribute is $24 \times 3 = 72$.
Notice that it is possible to arrange the edge cubes to show any total from 48 to 72 inclusive.
This is because if we want the total to be $48 + k$ where k is any integer from 0 to 24 inclusive, we simply need to arrange exactly k of the edge cubes to show a total of 3.
Each of the face cubes shows a total of 1 or a total of 2. There are 24 of them, so the total showing on the face cubes is at least 24 and at most $24 \times 2 = 48$.
Similar to the reasoning for the edge cubes, every integer from 24 to 48 inclusive is a possibility for the total showing on the face cubes.
Therefore, the smallest possible total is $32 + 48 + 24 = 104$, and the largest possible total is $32 + 72 + 48 = 152$.
By earlier reasoning, every integer between 104 and 152 inclusive is a possibility, so the number of possibilities is $152 - 103 = 49$.

Answer: 49

24. The sum of the positive integers from 1 to 9 inclusive is 45. Since each row sum must be the same, this means each row must be one third of 45 which is 15.
In the second row, the first two cells contain the integers 1 and 8, so the third must contain $15 - 8 - 1 = 6$.
The integers that still need to be placed are 2, 3, 4, 5, 7, and 9.
Notice that $15 - 9 = 6$ and that 2 and 4 are the only two integers in this list that have a sum of 6.

Therefore, the integers 9, 4, and 2 must be in one of the remaining rows and the integers 3, 5, and 7 must be in the other.
In fact, if the integers are placed in such a way that all conditions are satisfied and the top and bottom rows are exchanged, all properties will still be satisfied and the column products will not have changed.
Therefore, we can assume that 2, 4, and 9 are in the top row.
This gives six possible ways to fill in the top two rows. They are shown below.

2	4	9
1	8	6

2	9	4
1	8	6

4	2	9
1	8	6

4	9	2
1	8	6

9	2	4
1	8	6

9	4	2
1	8	6

The digits that remain to be placed are 3, 5, and 7.
We will first focus on the first partially filled in grid in which the first row contains 2, 4, and 9 from left to right.
The third column already contains 9 and 6. The integer to be placed in the third column in the bottom row is at least 3, so this means the third column product is at least $9 \times 6 \times 3 = 162$.
By similar reasoning, the first column product is at most $2 \times 1 \times 7 = 14$.
Therefore, no matter how the remaining integers are placed, the difference between the largest and smallest column product will be at least $162 - 14 = 148$, which is greater than 40.
Therefore, the 2, 4, and 9 cannot be placed in that order.
In the second grid, the second column product is at least $9 \times 8 \times 3 = 216$ and the first column product is at most $2 \times 1 \times 7 = 14$, so the difference between the largest and smallest column product is at least $216 - 14 = 202$ which is greater than 40, so the 2, 4, and 9 cannot be placed in the order 2, 9, 4, either.
By similar reasoning, the third and fourth possibilities above can be eliminated.
Therefore, the top row must contain, from left to right, either 9, 2, and 4, or 9, 4, and 2.
If the 5 is placed in the third column of the grid below,

9	2	4
1	8	6

then the third column product will be $4 \times 6 \times 5 = 120$.
This would mean that the largest the first column product can be is $9 \times 1 \times 7 = 63$.
This would lead to the largest and smallest column products having a difference greater than 40, so 5 cannot be placed in the third column.
If 7 is placed in the third column, the difference between the largest and smallest column products would be even larger, so the 3 must be placed in the third column.
By similar reasoning, the 3 must be placed in the second column of the last grid.
We have now reduced to four possible ways to place the integers, and they are shown below with the column products given below each column.

9	2	4
1	8	6
5	7	3
45	*112*	*72*

9	2	4
1	8	6
7	5	3
63	*80*	*72*

9	4	2
1	8	6
5	3	7
45	*96*	*84*

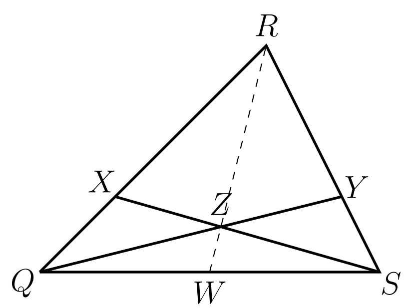
9	4	2
1	8	6
7	3	5
63	*96*	*60*

The column products in the first grid are 45, 112, and 72. Since 45 and 112 differ by more than 40, this grid does not satisfy the conditions.
The column products in the second grid are 63, 80, and 72.
The difference between the largest and smallest column product is $80 - 63 = 17$ which is less than 40, so this grid satisfies all of the conditions.
Its largest column product is 80, so 80 is one of the possibilities.
The column products in the third grid are 45, 96, and 84. Since 96 and 45 differ by more than 40, this grid does not satisfy the conditions.
The column products in the fourth grid are 63, 96, and 60.
The difference between the largest and smallest column product is $96 - 60 = 36$ which is less than 40, so this grid satisfies all of the conditions.
Its largest column product is 96, so 96 is another possibility.
We have shown that there are exactly two ways to arrange the integers so that the conditions are satisfied, and the largest column products in these two arrangements are 80 and 96.

ANSWER: 80, 96

25. The solution will use two facts. The first is a general fact about triangles.

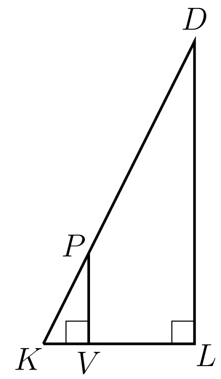
Fact 1: Suppose $\triangle QRS$ has X on QR and Y on SR so that $XR = 2QX$ and $YR = 2SY$. If Z is the point of intersection of QY and SX and W is the point where the extension of RZ intersects QS, then $WR = 5WZ$.



The second fact that we will use is specific to the question.

Fact 2: The point M is on the line segment connecting D to E so that $MD = 2EM$.
(Note: this justifies the fact that EH and CM intersect as implied by the problem statement.)

We will answer the question before proving the two facts.
To start, we will show that the altitude of tetrahedron $ABCD$ is 5 times that of $EBCP$.
Let K be the point where the line through DP intersects $\triangle ABC$ and let L be the point in $\triangle ABC$ so that DL is perpendicular to $\triangle ABC$, and let V be the point on $\triangle ABC$ so that PV is perpendicular to $\triangle ABC$.
By construction, DL is the altitude of $ABCD$ from A to $\triangle ABC$ and PV is the altitude of $EBCP$ from P to $\triangle EBC$.



By Fact 2, M is on DE so that $MD = 2EM$ and H is on DC so that $HD = 2CH$ by assumption.
By Fact 1, $KD = 5KP$.
Since $\triangle KPV$ and $\triangle KDL$ share an angle at K and both have a right angle, the triangles are similar.
This means $\dfrac{PV}{DL} = \dfrac{KP}{KD} = \dfrac{1}{5}$.
Thus, the length of the altitude of $ABCD$ is five times that of $EBCP$.
Since E is the midpoint of AB, the area of $\triangle EBC$ is half that of $\triangle ABC$.
Thus, the base area of tetrahedron $EBCP$ is half that of tetrahedron $ABCD$, while the heights are in ratio $1 : 5$, which means that the volumes are in the ratio $1 : 10$.

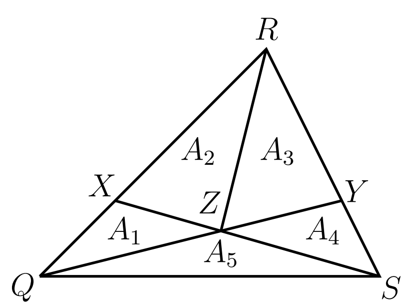
Now for the proofs of the facts.

Proof of Fact 1

We will first show that the height of $\triangle QZS$ from Z is one fifth the height of $\triangle QRS$ from R.
Suppose $\triangle QZS$ has height h_1 from Z and $\triangle QRS$ has height h_2 from R. The first goal is to show that $\dfrac{h_1}{h_2} = \dfrac{1}{5}$.
The area of $\triangle QZS$ is $\dfrac{1}{2}(QS)h_1$ and the area of $\triangle QRS$ is $\dfrac{1}{2}(QS)h_2$.
Thus, if we take the ratio of their areas we get

$$\frac{\frac{1}{2}(QS)h_1}{\frac{1}{2}(QS)h_2} = \frac{h_1}{h_2}$$

and so the ratio of their heights is the same as the ratio of their areas.
This means we can show that $\dfrac{h_1}{h_2} = \dfrac{1}{5}$ by showing that the area of $\triangle QRS$ is 5 times the area of $\triangle QZS$.
To do this, draw a line from R to Z and label the areas of $\triangle QXZ$, $\triangle RXZ$, $\triangle RYZ$, $\triangle SYZ$, and $\triangle QZS$ by A_1, A_2, A_3, A_4 and A_5, respectively.



We will now make several observations about the areas A_1 through A_5.
Since $\triangle QXZ$ and $\triangle RXZ$ have the same altitude from Z to QR and their bases satisfy $XR = 2QX$, we get that

$$A_2 = 2A_1 \tag{1}$$

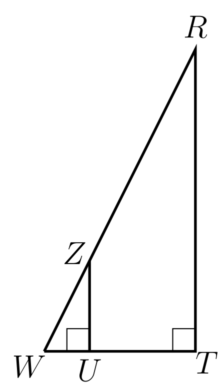
By similar reasoning, we also get the equations

$$A_3 = 2A_4 \tag{2}$$

$$A_2 + A_3 + A_4 = 2(A_1 + A_5) = 2A_1 + 2A_5 \tag{3}$$

$$A_1 + A_2 + A_3 = 2(A_4 + A_5) = 2A_4 + 2A_5 \tag{4}$$

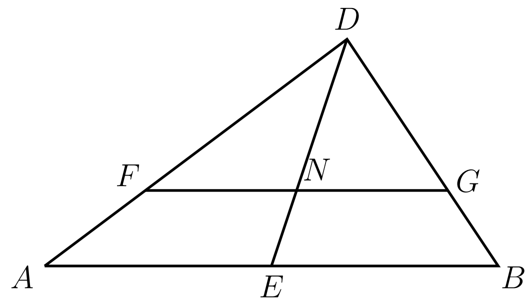
Substituting Equation (1) into Equation (3) gives $A_2 + A_3 + A_4 = A_2 + 2A_5$ which simplifies to $A_3 + A_4 = 2A_5$.
Substituting Equation (2) into Equation (4) gives $A_1 + A_2 + A_3 = A_3 + 2A_5$ which simplifies to $A_1 + A_2 = 2A_5$.
Adding $A_3 + A_4 = 2A_5$ and $A_1 + A_2 = 2A_5$ gives $A_1 + A_2 + A_3 + A_4 = 4A_5$.
Adding A_5 to both sides gives $A_1 + A_2 + A_3 + A_4 + A_5 = 5A_5$, which exactly says that the area of $\triangle QRS$ is five times that of $\triangle QZS$.
We will now let T be the point on QS so that RT is perpendicular to QS and let U be on WT so that ZU is perpendicular to RW.



Note that ZU is the height of $\triangle QZS$ from Z and RT is the height of $\triangle QRS$ from R, so we know that $RT = 5ZU$.
Since $\angle WUZ = \angle WTR = 90°$ and $\angle ZWU = \angle RWT$, $\triangle ZWU$ is similar to $\triangle RWT$.
Since $\dfrac{ZU}{RT} = \dfrac{1}{5}$, we get that $\dfrac{WZ}{WR} = \dfrac{1}{5}$, which proves the fact.

Proof of Fact 2

Consider $\triangle ADB$ and $\triangle FDG$. It is given that $\dfrac{FD}{AD} = \dfrac{2}{3} = \dfrac{GD}{BD}$, and since the triangles have a common angle at D, they are similar by side-angle-side similarity.
Let N be the point at which DE intersects FG.



Since $\triangle ADB$ is similar to $\triangle FDG$, $\angle DFN = \angle DAE$.
We also have that $\angle FDN = \angle ADE$ since they are the same angle, which implies that $\triangle FDN$ is similar to $\triangle ADE$ since if two triangles have two angles in common, they must have three angles in common.
Since $\frac{FD}{AD} = \frac{2}{3}$, similarity implies $\frac{FN}{AE} = \frac{2}{3}$.
By similar reasoning, $\frac{GN}{BE} = \frac{2}{3}$.
Rearranging these two equations leads to $FN = \frac{2}{3}AE$ and $GN = \frac{2}{3}BE$, but E is the midpoint of AB, so $AE = BE$, and hence, $FN = GN$.
This means N is the midpoint of FG, so $M = N$, which shows that M is on DE since N is on DE.
Finally, since $\triangle FDM$ and $\triangle ADE$ are similar, that $\frac{MD}{ED} = \frac{FD}{AD} = \frac{2}{3}$, which means that M is on DE so that $MD = 2EM$.

Answer: $\frac{1}{10}$

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each Relay are written as if the value of t is not initially known, and then t is substituted at the end.)

0. (a) Evaluating, $\dfrac{2+5\times 5}{3} = \dfrac{2+25}{3} = \dfrac{27}{3} = 9$.

   (b) The area of a triangle with base $2t$ and height $2t-6$ is $\frac{1}{2}(2t)(2t-6)$ or $t(2t-6)$.

   The answer to (a) is 9, so $t=9$ which means $t(2t-6) = 9(12) = 108$.

   (c) Since $\triangle ABC$ is isosceles with $AB = BC$, it is also true that $\angle BCA = \angle BAC$.
   The angles in a triangle add to 180°, so

$$\begin{aligned}180^\circ &= \angle ABC + \angle BAC + \angle BCA \\ &= \angle ABC + 2\angle BAC \\ &= t^\circ + 2\angle BAC\end{aligned}$$

   The answer to (b) is 108, so $t = 108$. Therefore,

$$\angle BAC = \frac{1}{2}(180^\circ - t^\circ) = \frac{1}{2}(180^\circ - 108^\circ) = \frac{1}{2}(72^\circ) = 36^\circ.$$

ANSWER: 9, 108, 36°

1. (a) Since 33 is positive, $\dfrac{\square}{11} < \dfrac{2}{3}$ implies

$$33\left(\frac{\square}{11}\right) < 33\left(\frac{2}{3}\right)$$

   which simplifies to $3\times\square < 22$.
   The largest multiple of 3 that is less than 22 is 3×7, so this means the number in the box cannot be larger than 7.
   Indeed, $\dfrac{7}{11} = \dfrac{21}{33}$ is less than $\dfrac{2}{3} = \dfrac{22}{33}$, so the answer is 7.

   (b) Rearranging the equation, we have $2x = -t-9$ or $x = \dfrac{-t-9}{2}$, so

$$x + 4 = \frac{-t-9}{2} + 4 = \frac{-t-9+8}{2} = \frac{-t-1}{2}$$

   Substituting $t = 7$ into this equation gives $x + 4 = \dfrac{-7-1}{2} = -4$.

   (c) The vertices of the triangle are the origin and the points at which the line intersects the x- and y-axes.
   When $y = 0$, we get $0 = tx + 6$ so $x = \dfrac{-6}{t}$.
   When $x = 0$, we get $y = t(0) + 6$ so $y = 6$.
   The triangle is right-angled at the origin, its height is 6, and its base the distance from $\dfrac{-6}{t}$ to the origin.
   If t is negative, then the base is $\dfrac{-6}{t}$, and if t is positive, then the base is $\dfrac{6}{t}$.
   Therefore, if t is negative, then the area of the triangle is

$$\frac{1}{2}\times 6\times\frac{-6}{t} = \frac{-18}{t}$$

and if t is positive, then the area of the triangle is

$$\frac{1}{2} \times 6 \times \frac{6}{t} = \frac{18}{t}$$

Since $t = -4$ which is negative, the area of the triangle is $\dfrac{-18}{-4} = \dfrac{9}{2}$.

Answer: $\left(7, -4, \frac{9}{2}\right)$

2. (a) The prime numbers between 10 and 30 are 11, 13, 17, 19, 23, and 29, which means $x = 6$.
Factoring the numerator in $\dfrac{x^2 - 4}{x + 2}$ gives $\dfrac{(x-2)(x+2)}{x+2}$ which is equal to $x - 2$ as long as $x \neq 2$.
Since $x = 6$, the answer is $6 - 2 = 4$.

(b) Let A be the number of beans that Alida has, B be the number of beans that Bono has, and C be the number of beans that Cate has.
The given information translates to

$$\begin{aligned} A + B &= 6t + 3 \\ A + C &= 4t + 5 \\ B + C &= 6t \end{aligned}$$

Adding these three equations together gives $2(A + B + C) = 16t + 8$, which implies $A + B + C = 8t + 4$.
Subtracting $A + C = 4t + 5$ from $A + B + C = 8t + 4$ gives

$$B = (A + B + C) - (A + C) = (8t + 4) - (4t + 5) = 4t - 1$$

Substituting $t = 4$ into $B = 4t - 1$ gives $B = 4(4) - 1 = 15$.

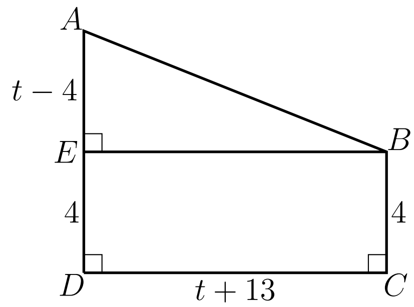
(c) Suppose x is the real number such that $x^2 - tx + 36 = 0$ and $x^2 - 8x + t = 0$.
Therefore, we get that $x^2 - tx + 36 = x^2 - 8x + t$ which can be rearranged to get $36 - t = tx - 8x = (t - 8)x$.
Dividing by $t - 8$ gives $x = \dfrac{36 - t}{t - 8}$.
Substituting $t = 15$ gives $x = \dfrac{36 - 15}{15 - 8} = \dfrac{21}{7} = 3$.

Answer: $(4, 15, 3)$

3. (a) A cube with edge length x has six square faces with area x^2, so its surface area is $6x^2$.
This means $6x^2 = 1014$ so $x^2 = 169$.
Since x is an edge length, it is positive, so $x = \sqrt{169} = 13$.

(b) Multiplying through by $3(t + x)$ gives $3(5 + x) = 2(t + x)$.
Expanding gives $15 + 3x = 2t + 2x$ which can be rearranged to $x = 2t - 15$.
Substituting $t = 13$ gives $x = 2(13) - 15 = 11$.

(c) Let E be on AD so that BE is perpendicular to AE.



Quadrilateral $BCDE$ has three right angles, which means that it must have four right angles.
This means that quadrilateral $BCDE$ is a rectangle, so $BE = CD = t + 13$.
We also have that $ED = BC = 4$, so $AE = AD - DE = t - 4$.
By the Pythagorean theorem applied to $\triangle AEB$, we get

$$AB^2 = (t-4)^2 + (t+13)^2 = 2t^2 + 18t + 185$$

Since AB is positive, $AB = \sqrt{2t^2 + 18t + 185}$.
Since $t = 11$,

$$AB = \sqrt{2(121) + 18(11) + 185} = \sqrt{242 + 198 + 185} = \sqrt{625} = 25$$

Answer: $(13, 11, 25)$