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2022 Canadian Senior Mathematics Contest

Wednesday, November 16, 2022
(in North America and South America)

Thursday, November 17, 2022
(outside of North America and South America)

Solutions

Part A

1. Since $2 \cdot 2 \cdot 2 \cdot 2 = 16$, then $2^4 = 16$ and so $r = 4$.
   Since $5 \cdot 5 = 25$, then $5^2 = 25$ and so $s = 2$.
   Therefore, $r + s = 4 + 2 = 6$.

   Answer: 6

2. *Solution 1*
   Since $\dfrac{x+y}{2} = 5$, then $x + y = 2 \cdot 5 = 10$. Since $\dfrac{x-y}{2} = 2$, then $x - y = 2 \cdot 2 = 4$.
   Therefore, $x^2 - y^2 = (x+y)(x-y) = 10 \cdot 4 = 40$.

   Solution 2
   Since $\dfrac{x+y}{2} = 5$ and $\dfrac{x-y}{2} = 2$, then $\dfrac{x+y}{2} + \dfrac{x-y}{2} = 5 + 2$ which simplifies to $x = 7$.
   Also, $\dfrac{x+y}{2} - \dfrac{x-y}{2} = 5 - 2$ which simplifies to $y = 3$.
   Therefore, $x^2 - y^2 = 7^2 - 3^2 = 49 - 9 = 40$.

   Answer: 40

3. Suppose that the two integers are m and n.
   We are told that $m + n = 60$ and $\operatorname{lcm}(m, n) = 273$.
   We note that $273 = 3 \cdot 91 = 3 \cdot 7 \cdot 13$.
   Since 273 is a multiple of each of m and n, this means that m and n are both divisors of 273.
   The divisors of 273 are 1, 3, 7, 13, 21, 39, 91, 273.
   Of these, the only pair that has a sum of 60 is 21 and 39.
   Thus, the two integers are 21 and 39.
   We note that since $21 = 3 \cdot 7$ and $39 = 3 \cdot 13$, then an integer is a common multiple of 21 and 39 exactly when it has prime factors of 3, 7 and 13, so the smallest such integer is $3 \cdot 7 \cdot 13 = 273$ which makes $\operatorname{lcm}(21, 39) = 273$.

   Answer: 21 and 39

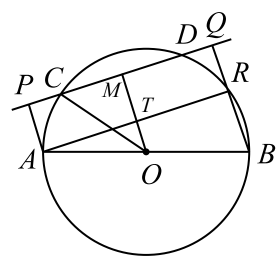
4. By the Pythagorean Theorem,
   $$CD^2 = AC^2 - AD^2 = 625^2 - 600^2 = 390\,625 - 360\,000 = 30\,625 = 175^2$$
   Since $CD > 0$, we get $CD = 175$.
   Let $\angle DAC = \theta$. Thus, $\angle BAC = 2\angle DAC = 2\theta$.
   Looking in $\triangle ADC$, we see that $\sin\theta = \dfrac{CD}{AC} = \dfrac{175}{625} = \dfrac{7}{25}$ and $\cos\theta = \dfrac{AD}{AC} = \dfrac{600}{625} = \dfrac{24}{25}$.
   Thus,
   $$\sin 2\theta = 2\sin\theta\cos\theta = 2 \cdot \frac{7}{25} \cdot \frac{24}{25} = \frac{336}{625}$$
   Since $\sin(\angle BAC) = \dfrac{BC}{AC}$, then
   $$BC = AC\sin(\angle BAC) = 625\sin 2\theta = 625 \cdot \frac{336}{625}$$
   and so $BC = 336$.

   Answer: 336

5. Since the circle has diameter $2\sqrt{19}$, it has radius $\sqrt{19}$.
Let $AP = 2x$. Thus, $BQ = 2AP = 4x$.
Let $CD = 2y$ and let M be the midpoint of CD. Thus, $CM = MD = y$.
Join A to R. Since AB is a diameter, then $\angle ARB = 90^\circ$.
Since quadrilateral $PQRA$ has right angles at P, Q and R, then it must have four right angles, and so it is a rectangle.
Thus, $AR = PQ = PC + CD + DQ = 1 + 2y + 1 = 2y + 2$ and $RQ = AP = 2x$.
Since $BQ = 4x$, then $RB = BQ - RQ = 4x - 2x = 2x$.
By the Pythagorean Theorem in $\triangle ARB$, we obtain

$$\begin{aligned} AR^2 + RB^2 &= AB^2 \\ (2y+2)^2 + (2x)^2 &= (2\sqrt{19})^2 \\ (y+1)^2 + x^2 &= 19 \end{aligned}$$

Join O to M.



Since M is the midpoint of chord CD, then OM is perpendicular to CD.
Since AR is parallel to CD (opposite sides in rectangle $PQRA$), then OM is also perpendicular to AR, crossing AR at T.
Consider $\triangle ATO$ and $\triangle ARB$. These triangles are similar because they share a common angle at A and each is right-angled (at T and R, respectively).
Since $\dfrac{AO}{AB} = \dfrac{1}{2}$ (radius and diameter), then $\dfrac{TO}{RB} = \dfrac{1}{2}$ and so $TO = \dfrac{1}{2} \cdot RB = x$.
Therefore, $OM = MT + TO = 2x + x = 3x$. This is because $MT = PA$ since MT is parallel to side PA of rectangle $PQRA$ and has its endpoints on opposite sides of the rectangle.
By the Pythagorean Theorem in $\triangle OMC$, we have

$$\begin{aligned} CM^2 + OM^2 &= CO^2 \\ y^2 + (3x)^2 &= (\sqrt{19})^2 \qquad \text{(since } CO \text{ is a radius)} \\ y^2 + 9x^2 &= 19 \end{aligned}$$

Since $(y+1)^2 + x^2 = 19$, then multiplying by 9 gives $9(y+1)^2 + 9x^2 = 171$.
Subtracting $y^2 + 9x^2 = 19$ gives

$$\begin{aligned} 9(y^2 + 2y + 1) - y^2 &= 171 - 19 \\ 8y^2 + 18y - 143 &= 0 \\ (4y - 13)(2y + 11) &= 0 \end{aligned}$$

Since $y > 0$, then $y = \dfrac{13}{4}$. Therefore, $x^2 = 19 - (y+1)^2 = 19 - \left(\dfrac{17}{4}\right)^2 = \dfrac{304}{16} - \dfrac{289}{16} = \dfrac{15}{16}$.
Since $x > 0$, then $x = \dfrac{\sqrt{15}}{4}$. Finally, $AP = 2x = \dfrac{\sqrt{15}}{2}$.

Answer: $\dfrac{\sqrt{15}}{2}$

6. We use R to represent a red marble, B to represent blue, and G to represent green.

   Since there are 15 marbles of which 3 are red, 5 are blue, and 7 are green, there are $\dfrac{15!}{3!5!7!}$ orders in which the 15 marbles can be removed.
   Since red is the first colour to have 0 remaining, the last of the 15 marbles removed must be B or G. We look at these two cases.

   Suppose that the final marble is B.

   The remaining 4 B's are thus placed among the first 14 marbles. There are $\dbinom{14}{4}$ ways in which this can be done.
   This leaves 10 open spaces. The last of these 10 marbles cannot be R, otherwise the last G would be chosen before the last R. Thus, the last of these 10 marbles is G.

   The remaining 6 G's are thus placed among the remaining 9 spaces. There are $\dbinom{9}{6}$ ways in which this can be done.
   The 3 R's are then placed in the remaining 3 open spaces.
   Therefore, there are $\dbinom{14}{4}\dbinom{9}{6}$ orders when the final marble is B.

   Suppose that the final marble is G.

   The remaining 6 G's are thus placed among the first 14 marbles. There are $\dbinom{14}{6}$ ways in which this can be done.
   This leaves 8 open spaces. The last of these 8 marbles cannot be R, otherwise the last B would be chosen before the last R. Thus, the last of these 8 marbles is B.

   The remaining 4 B's are thus placed among the remaining 7 spaces. There are $\dbinom{7}{4}$ ways in which this can be done.
   The 3 R's are then placed in the remaining 3 open spaces.
   Therefore, there are $\dbinom{14}{6}\dbinom{7}{4}$ orders when the final marble is G.

   Thus, the total number of ways in which the first colour with 0 remaining is red equals $\dbinom{14}{4}\dbinom{9}{6}+\dbinom{14}{6}\dbinom{7}{4}$.
   This means that the desired probability, p, is

$$
\begin{aligned}
p = \frac{\dbinom{14}{4}\dbinom{9}{6}+\dbinom{14}{6}\dbinom{7}{4}}{\dfrac{15!}{3!5!7!}} &= \frac{\dfrac{14!}{4!10!}\,\dfrac{9!}{6!3!}+\dfrac{14!}{6!8!}\,\dfrac{7!}{4!3!}}{\dfrac{15!}{3!5!7!}} \\
&= \frac{14!}{15!}\,\frac{9!}{10!}\,\frac{5!}{4!}\,\frac{7!}{6!}\,\frac{3!}{3!}+\frac{14!}{15!}\,\frac{7!}{8!}\,\frac{7!}{6!}\,\frac{5!}{4!}\,\frac{3!}{3!} \\
&= \frac{1}{15}\cdot\frac{1}{10}\cdot 5\cdot 7\cdot 1+\frac{1}{15}\cdot\frac{1}{8}\cdot 7\cdot 5\cdot 1 \\
&= \frac{7}{30}+\frac{7}{24}=\frac{28}{120}+\frac{35}{120}=\frac{63}{120}=\frac{21}{40}
\end{aligned}
$$

ANSWER: $\dfrac{21}{40}$

Part B

1. (a) Points A and B are the x-intercepts of the parabola.
To find their coordinates, we set $y = 0$ to obtain the equation $0 = -x^2 + 16$, which gives $x^2 = 16$ and so $x = \pm 4$.
Therefore, A has coordinates $(-4, 0)$ and B has coordinates $(4, 0)$.

(b) We start by finding the coordinates of M and N.
Since these points lie on the horizontal line with equation $y = 7$ and on the parabola with equation $y = -x^2 + 16$, we equate expressions for y to obtain $7 = -x^2 + 16$ which gives $x^2 = 9$ and so $x = \pm 3$.
Thus, M has coordinates $(-3, 7)$ and N has coordinates $(3, 7)$.
Trapezoid $MNBA$ has parallel, horizontal sides MN and AB.
Here, $MN = 3 - (-3) = 6$ and $AB = 4 - (-4) = 8$.
Also, the height of $MNBA$ is 7 since MN lies along $y = 7$ and AB lies along $y = 0$.
Therefore, the area of $MNBA$ is $\frac{1}{2}(6 + 8) \cdot 7 = 49$.

(c) The origin, O, has coordinates $(0, 0)$.
The vertex, V, of the parabola has coordinates $(0, 16)$, since it lies on the y-axis and substituting $x = 0$ into the equation $y = -x^2 + 16$ gives $y = 16$.
When we set $y = -33$ in the equation of the parabola $y = -x^2 + 16$, we obtain $x^2 = 49$ and so $x = \pm 7$.
The area of quadrilateral $VPOQ$ can be found by adding the areas of $\triangle VOP$ and $\triangle VOQ$.
Each of these triangles can be thought of as having a vertical base VO of length 16, and a horizontal height.
The length of each horizontal height is 7, since the distance from each of P and Q to the y-axis is 7.
Therefore, the area of $VPOQ$ is $2 \cdot \frac{1}{2} \cdot 16 \cdot 7 = 112$.

2. (a) We begin by noting that $\sqrt{a^2 + a} = \frac{2}{3}$ is equivalent to $a^2 + a = \frac{4}{9}$.
Multiplying this equation by 9 and re-arranging, we obtain the equivalent equation $9a^2 + 9a - 4 = 0$.
Factoring, we obtain $(3a - 1)(3a + 4) = 0$ and so $a = \frac{1}{3}$ or $a = -\frac{4}{3}$.
Since $a > 0$, then $a = \frac{1}{3}$.
Substituting into the original equation, we obtain

$$\sqrt{a^2 + a} = \sqrt{\left(\tfrac{1}{3}\right)^2 + \tfrac{1}{3}} = \sqrt{\tfrac{1}{9} + \tfrac{1}{3}} = \sqrt{\tfrac{4}{9}} = \tfrac{2}{3}$$

and so $a = \frac{1}{3}$ is the only solution to the equation when $a > 0$.

(b) Expanding and simplifying,

$$(m + \tfrac{1}{2})^2 + (m + \tfrac{1}{2}) = m^2 + m + \tfrac{1}{4} + m + \tfrac{1}{2} = m^2 + 2m + \tfrac{3}{4}$$

Since m is a positive integer, then $m^2 + 2m + \frac{3}{4} > m^2$.
Also, $m^2 + 2m + \frac{3}{4} < m^2 + 2m + 1 = (m + 1)^2$.
Thus, $m^2 < m^2 + 2m + \frac{3}{4} < (m + 1)^2$ which means that the closest perfect square is either m^2 or $(m + 1)^2$.
The difference between $m^2 + 2m + \frac{3}{4}$ and m^2 is $2m + \frac{3}{4}$.
The difference between $m^2 + 2m + \frac{3}{4}$ and $(m + 1)^2$ is $\frac{1}{4}$.
Since m is a positive integer, $2m + \frac{3}{4} > \frac{1}{4}$, and so $(m + \frac{1}{2})^2 + (m + \frac{1}{2})$ is closest to $(m + 1)^2$ and their difference is $\frac{1}{4}$.

(c) A fact that will be important throughout this solution is that the expresssion $c + \sqrt{c}$ increases as c increases.
In other words, when $0 < c < d$, then $c + \sqrt{c} < d + \sqrt{d}$.
This is true since when $0 < c < d$, then $\sqrt{c} < \sqrt{d}$ and so $c + \sqrt{c} < d + \sqrt{d}$.

Before proving a general result, we determine the number of positive integers c that satisfy the inequality for each of $n = 1$, $n = 2$, and $n = 3$.

When $n = 1$, the inequality is $1 < \sqrt{c + \sqrt{c}} < 2$.
Since each part is positive, we can square each part and obtain the equivalent inequality $1 < c + \sqrt{c} < 4$.
We see that $1 + \sqrt{1} = 2$ and $3 < 2 + \sqrt{2} < 4$ (because $\sqrt{2}$ is between 1 and 2), and $3 + \sqrt{3} > 4$ (because $\sqrt{3} > 1$).
Thus, when $n = 1$, there are 2 values of c that work.

When $n = 2$, the inequality is $2 < \sqrt{c + \sqrt{c}} < 3$ which is equivalent to $4 < c + \sqrt{c} < 9$.
We note that $2 + \sqrt{2} < 4$ (because $\sqrt{2} < 2$), that $4 < 3 + \sqrt{3} < 5$ (because $\sqrt{3}$ is between 1 and 2), that $8 < 6 + \sqrt{6} < 9$ (because $\sqrt{6}$ is between 2 and 3), and that $9 < 7 + \sqrt{7}$ (because $\sqrt{7} > 2$).
Since $c + \sqrt{c}$ is increasing, the inequality $4 < c + \sqrt{c} < 9$ is true for exactly the positive integers $c = 3, 4, 5, 6$, since it is true for $c = 3$ and $c = 6$ and so is true for all integers in between.

When $n = 3$, the inequality is $3 < \sqrt{c + \sqrt{c}} < 4$ which is equivalent to $9 < c + \sqrt{c} < 16$.
Using similar reasoning, we obtain that this inequality is true for the positive integers $c = 7, 8, 9, 10, 11, 12$.

Therefore, for $n = 1, 2, 3$, we obtain 2, 4 and 6 solutions.
Based on these values of n, we guess that for a general value of n, there are $2n$ values of c that work. Also, for each n the largest value of c that works appears to be $c = n(n + 1)$ and the smallest value of c that works appears to be $c = n(n - 1) + 1$. (Check to see that these conjectures match the specific cases above.)

From earlier, we know that the inequality $n < \sqrt{c + \sqrt{c}} < n + 1$ is equivalent to the inequality $n^2 < c + \sqrt{c} < n^2 + 2n + 1$.
Our strategy now is to show that the inequality

$$n^2 < c + \sqrt{c} < n^2 + 2n + 1$$

(I) is false when $c = n(n - 1)$, (II) is true when $c = n(n - 1) + 1$, (III) is true when $c = n(n + 1)$, and (IV) is false when $c = n(n + 1) + 1$.
Since the expression $c + \sqrt{c}$ is increasing, these facts will show that the positive integers c for which the inequality is true are exactly the values of c with $n(n-1)+1 \leq c \leq n(n+1)$, of which there are

$$n(n + 1) - (n(n - 1) + 1) + 1 = n^2 + n - (n^2 - n + 1) + 1 = 2n$$

and so the number of c will be even, as required.

Step (I)
We note that when $n \geq 1$, we have $n(n - 1) = n^2 - n < n^2$ and so $\sqrt{n(n - 1)} < \sqrt{n^2} = n$.
Thus, when $c = n(n - 1)$, we see that

$$c + \sqrt{c} = n(n - 1) + \sqrt{n(n - 1)} = n^2 - n + \sqrt{n(n - 1)} < n^2 - n + n = n^2$$

and so the inequality $n^2 < c + \sqrt{c} < n^2 + 2n + 1$ is false.

Step (II)

Next, we note that when $n \geq 1$, we have $n(n-1) + 1 = n^2 - n + 1 > n^2 - 2n + 1$ and so $\sqrt{n(n-1)+1} > \sqrt{n^2 - 2n + 1} = n - 1$.

Also, when $n \geq 1$, we have $n(n-1)+1 = n^2 - n + 1 \leq n^2$ and so $\sqrt{n(n-1)+1} \leq \sqrt{n^2} = n$.

Thus, when $c = n(n-1) + 1$, we see that

$$c+\sqrt{c} = n(n-1)+1+\sqrt{n(n-1)+1} = n^2-n+1+\sqrt{n(n-1)+1} > n^2-n+1+(n-1) = n^2$$

Also, when $c = n(n-1) + 1$, we see that

$$c + \sqrt{c} = n(n-1) + 1 + \sqrt{n(n-1)+1} \leq n^2 - n + 1 + n = n^2 + 1 < n^2 + 2n + 1$$

Thus, the inequality $n^2 < c + \sqrt{c} < n^2 + 2n + 1$ is true.

Connecting (I) and (II) to (III) and (IV)

We have shown that, for all integers $n \geq 1$,

$$n(n-1) + \sqrt{n(n-1)} < n^2$$

and

$$n^2 < (n(n-1)+1) + \sqrt{n(n-1)+1} < n^2 + 2n + 1$$

This means that, for all integers $m \geq 1$,

$$m(m-1) + \sqrt{m(m-1)} < m^2$$

and

$$m^2 < (m(m-1)+1) + \sqrt{m(m-1)+1} < (m+1)^2$$

Setting $m = n + 1$ and restricting $n \geq 1$ (and so $m \geq 2$), we obtain

$$n(n+1) + \sqrt{n(n+1)} < (n+1)^2 \qquad (*)$$

and

$$(n+1)^2 < (n(n+1)+1) + \sqrt{n(n+1)+1} < (n+2)^2$$

This last set of inequalities shows that $n^2 < c+\sqrt{c} < (n+1)^2$ is false when $c = n(n+1)+1$, which is item (IV).

Since $n(n+1) > n^2$, we can update $(*)$ to

$$n^2 < n(n+1) + \sqrt{n(n+1)} < (n+1)^2$$

which is item (III).

We have now shown the desired inequalities for $c = n(n-1)$, $c = n(n-1)+1$, $c = n(n+1)$, and $c = n(n+1) + 1$, which means that the number of positive integers c that satisfy the inequality for a given n is $2n$, which is even, as required.

3. (a) We note that if $(a, b, c, d) = (A, B, C, D)$ satisfies the equation $a - b = c - d$ (that is, if $A - B = C - D$), then so do $(a, b, c, d) = (B, A, D, C)$ and $(a, b, c, d) = (C, D, A, B)$ and $(a, b, c, d) = (D, C, B, A)$ since

$$B - A = D - C \qquad C - D = A - B \qquad D - C = B - A$$

Let $g(n)$ be the number of quadruples (a, b, c, d) of distinct integers from S_n for which $a - b = c - d$ with $a < b$ and $c < d$ and $a < c$.
Then $f(n) = 4g(n)$ since each of the quadruples counted by $g(n)$ can be transformed into four quadruples counted by $f(n)$ by using the four arrangements above.
We note that every quadruple counted by $f(n)$ has either $a < b$ and $c < d$ or $a > b$ and $c > d$ (because $a - b = c - d$ and so the signs of $a - b$ and $c - d$ are the same) and has either $a < c$ or $a > c$ (because $a \neq c$).
Therefore, the quadruples counted by $f(n)$ can be grouped into sets of four quadruples exactly one of which satisfies $a < b$ and $c < d$ and $a < c$.

To determine the value of $f(6)$, we first determine the value of $g(6)$.
To determine the value of $g(6)$, we look at the possible pairs (a, b) with $a < b$ and for each of these count the number of possible pairs (c, d).
We work through pairs (a, b) by increasing value of $b - a$, recalling that $c < d$ and $a < c$:

- $(a, b) = (1, 2)$: the possibilities for (c, d) are $(3, 4)$, $(4, 5)$, $(5, 6)$
- $(a, b) = (2, 3)$: the possibilities for (c, d) are $(4, 5)$, $(5, 6)$
- $(a, b) = (3, 4)$: the only possibility for (c, d) is $(5, 6)$
- $(a, b) = (1, 3)$: the possibilities for (c, d) are $(2, 4)$, $(4, 6)$
- $(a, b) = (2, 4)$: the only possibility for (c, d) is $(3, 5)$
- $(a, b) = (3, 5)$: the only possibility for (c, d) is $(4, 6)$
- $(a, b) = (1, 4)$: the possibilities for (c, d) are $(2, 5)$, $(3, 6)$
- $(a, b) = (2, 5)$: the only possibility for (c, d) is $(3, 6)$
- $(a, b) = (1, 5)$: the only possibility for (c, d) is $(2, 6)$

Since there are thus $3 + 2 + 1 + 2 + 1 + 1 + 2 + 1 + 1 = 14$ quadruples (a, b, c, d), we have $g(6) = 14$ and so $f(6) = 56$.

(b),(c) To answer parts (b) and (c), we determine an explicit expression for $g(n)$ (and hence for $f(n)$) in terms of n. With this information in hand, we can then answer the given questions.
Suppose that n is an even positive integer. We write $n = 2m$ for some positive integer $m \geq 1$.
To determine an expression for $g(2m)$, we count the number of quadruples (a, b, c, d) of distinct integers from S_n for which $a - b = c - d$ with $a < b$ and $c < d$ and $a < c$.
Since $a < b$ and $a < c$ and $b \neq c$, then either $b < c$ or $b > c$.
This gives us two cases for the ordering of a, b, c, d: first that $a < b < c < d$ and second that $a < c < b < d$. Note that $d > b$ in both cases because the "gap" between a and b is equal to the gap between c and d.

Case 1: $a < b < c < d$

Suppose that $b - a = d - c = t$ for some positive integer t.
Since $d - a = d - c + c - a > d - c + b - a = 2t$ and $d - a \leq 2m - 1$, then $2t \leq 2m - 1$, which means that $t \leq m - \frac{1}{2}$. Since t and m are integers, then $t \leq m - 1$.
We count the number of quadruples (a, b, c, d) with $a < b < c < d$ for each value of t from 1 to $m - 1$, inclusive.
Suppose that $t = 1$.
Here, $b = a + 1$ and the pair (c, d) can be as small as $(a + 2, a + 3)$ and as large as $(2m - 1, 2m)$.
For the pair $(a, b) = (a, a + 1)$, there are $2m - (a + 3) + 1 = 2m - a - 2$ pairs (c, d).
Since a can range from $a = 1$ to $a = 2m - 3$ (the latter of which gives the quadruple $(2m - 3, 2m - 2, 2m - 1, 2m)$), there is a total of

$$(2m - 3) + (2m - 4) + \cdots + 2 + 1 = \tfrac{1}{2}(2m - 3)(2m - 2)$$

quadruples (a, b, c, d) in the case $t = 1$.

More generally, suppose that $1 \leq t \leq m - 1$.
Here, $b = a + t$ and the pair (c, d) can be as small as $(a + t + 1, a + 2t + 1)$ and as large as $(2m - t, 2m)$.
For the pair $(a, b) = (a, a + t)$, there are thus $2m - (a + 2t + 1) + 1 = 2m - a - 2t$ pairs (c, d).
Since a can range from $a = 1$ to $a = 2m - 2t - 1$ (the latter of which gives the quadruple $(2m - 2t - 1, 2m - t - 1, 2m - t, 2m)$), there is a total of

$$(2m - 2t - 1) + (2m - 2t - 2) + \cdots + 2 + 1 = \tfrac{1}{2}(2m - 2t - 1)(2m - 2t)$$

quadruples (a, b, c, d) for each t with $1 \leq t \leq m - 1$.

We note that

$$\begin{aligned} \tfrac{1}{2}(2m - 2t - 1)(2m - 2t) &= (2m - 2t - 1)(m - t) \\ &= 2m^2 - 2mt - 2mt + 2t^2 - m + t \\ &= 2t^2 + t(-4m + 1) + (2m^2 - m) \end{aligned}$$

From the Useful Fact, when we add $2t^2$ for each t from 1 to $m - 1$, we obtain $2 \cdot \dfrac{(m - 1)m(2m - 1)}{6}$.
Adding $t(-4m + 1)$ for each t from 1 to $m - 1$ is equivalent to multiplying $(-4m + 1)$ by the sum of the integers from 1 to $m - 1$, which gives $(-4m + 1) \cdot \frac{1}{2}(m - 1)m$.
Adding $(2m^2 - m)$ for each t from 1 to $m - 1$ gives $(m - 1)(2m^2 - m)$.
Adding these partial sums together, we obtain the total number of triples in this case, which is

$$\begin{aligned} &2 \cdot \frac{(m - 1)m(2m - 1)}{6} + (-4m + 1) \cdot \tfrac{1}{2}(m - 1)m + (m - 1)(2m^2 - m) \\ &\qquad = m(m - 1)\left(\frac{2m - 1}{3} + \frac{-4m + 1}{2} + (2m - 1)\right) \\ &\qquad = m(m - 1)\left(\frac{4m - 2}{6} + \frac{-12m + 3}{6} + \frac{12m - 6}{6})\right) \\ &\qquad = m(m - 1) \cdot \frac{4m - 5}{6} \end{aligned}$$

Case 2: $a < c < b < d$
Suppose that $b - a = d - c = t$ for some positive integer t.
Since $2m \geq d \geq b + 1$, then $t = b - a \leq 2m - 1 - 1 = 2m - 2$. Since $a < c < b$, then $b \geq a + 2$, which means that $t \geq 2$.
Thus, the range of values for t is $2 \leq t \leq 2m - 1$.
For each pair of values (a, b), the value of c can, in theory, move between $a + 1$ and $b - 1$ with the restriction that $d = c + t$ must still be at most $2m$.

Our strategy is to work through the possible values of t, in each case starting with $a = 1$ to establish the possible quadruples in this case when $a = 1$, and then count the number of ways these relative positions for (a, b, c, d) can be slid until d reaches $2m$.

Suppose that $t = 2$ and $a = 1$. Here, $b = a + t = 3$. Since $a < c < b$, then $c = 2$ which gives $d = 4$.
The base quadruple $(a, b, c, d) = (1, 3, 2, 4)$ can have each component increased by 1 until we obtain the largest possible quadruple with this relative positioning, which is $(a, b, c, d) = (2m - 3, 2m - 1, 2m - 2, 2m)$.
There are $2m - 4 + 1 = 2m - 3$ quadruples in this case.

Suppose that $t = 3$ and $a = 1$. Here, $b = 4$ and so $c = 2$ or $c = 3$.
This gives the base quadruples $(a, b, c, d) = (1, 4, 2, 5)$ and $(a, b, c, d) = (1, 4, 3, 6)$.
There are $2m - 4$ and $2m - 5$ possible quadruples of these two forms.

Next, consider a general value of t with $2 \leq t \leq m$, along with $a = 1$.
Here, $b = t + 1$ and so $2 \leq c \leq t$.
This gives the base quadruples $(a, b, c, d) = (1, t + 1, c, c + t)$ for $2 \leq c \leq m$.
For each of these values of c, the corresponding value of d satisfies $d \leq 2m$, since $d = c + t \leq m + m = 2m$.
For a given value of c, there are $2m - (c + t) + 1 = 2m - c - t + 1$ such quadruples.
As c increases from 2 to t, we see that there are

$$(2m - t - 1) + (2m - t - 2) + \cdots + (2m - 2t + 2) + (2m - 2t + 1)$$

quadruples. There are $t - 1$ terms in this sum.
This sum can be re-written as

$$\begin{aligned}
&(2m - t - 1) + (2m - t - 2) + \cdots + (2m - 2t + 2) + (2m - 2t + 1) \\
&= ((2m - 2t) + (t - 1)) + ((2m - 2t) + (t - 2)) + \cdots + ((2m - 2t) + 2) + ((2m - 2t) + 1) \\
&= (t - 1)(2m - 2t) + (1 + 2 + \cdots + (t - 2) + (t - 1)) \\
&= (t - 1)(2m - 2t) + \tfrac{1}{2}t(t - 1) \\
&= 2mt - 2t^2 - 2m + 2t + \tfrac{1}{2}t^2 - \tfrac{1}{2}t \\
&= -\tfrac{3}{2}t^2 + t(2m + \tfrac{3}{2}) - 2m
\end{aligned}$$

When $t = 1$, this expression is equal to 0.
Thus, we can add these expressions for $t = 1$ to $t = m$ and obtain the same sum as adding from $t = 2$ to $t = m$.

This sum is thus

$$\begin{aligned}
&-\tfrac{3}{2}(1^2+2^2+\cdots+m^2)+(1+2+\cdots+m)(2m+\tfrac{3}{2})-2m\cdot m \\
&\quad= -\frac{3}{2}\cdot\frac{m(m+1)(2m+1)}{6}+\frac{4m+3}{2}\cdot\frac{m(m+1)}{2}-2m^2 \\
&\quad= -\frac{m(m+1)(2m+1)}{4}+\frac{m(m+1)(4m+3)}{4}-2m^2 \\
&\quad= \frac{m(m+1)(2m+2)}{4}-2m^2 \\
&\quad= \frac{m}{2}((m+1)^2-4m) \\
&\quad= \frac{m}{2}(m^2-2m+1) \\
&\quad= \frac{m(m-1)^2}{2}
\end{aligned}$$

Finally, consider a general value of t with $m+1 \le t \le 2m-2$.
Again, $(a,b,c,d)=(1,t+1,c,c+t)$ and $2 \le c \le t$, but $c+t \le 2m$ means that $c \le 2m-t \le 2(t-1)-t = t-2$ which is strictly less than the upper bound coming from the value of b.
In other words, when t is large enough, not all of the values of c between 2 and t actually produce admissible values of d.
As above, there are $2m-c-t+1$ triples that work for a given value of c, but the upper bound for c is different.
As c increases from 2 to $2m-t$, we see that there are

$$(2m-t-1)+(2m-t-2)+\cdots+2+1$$

quadruples, which equals $\frac{1}{2}(2m-t-1)(2m-t)$.
We need to add these expressions for $t=m+1$ to $t=2m-2$.
We substitute $s=2m-1-t$, which means that we need to add $\frac{1}{2}s(s+1)=\frac{1}{2}s^2+\frac{1}{2}s$ for $s=1$ to $s=m-2$.
This gives

$$\begin{aligned}
&\tfrac{1}{2}(1^2+2^2+\cdots(m-2)^2)+\tfrac{1}{2}(1+2+\cdots+(m-2)) \\
&\quad= \frac{1}{2}\cdot\frac{(m-2)(m-1)(2m-3)}{6}+\frac{1}{2}\frac{(m-2)(m-1)}{2} \\
&\quad= \frac{(m-2)(m-1)}{4}\left(\frac{2m-3}{3}+1\right) \\
&\quad= \frac{(m-2)(m-1)}{4}\cdot\frac{2m}{3} \\
&\quad= \frac{(m-2)(m-1)m}{6}
\end{aligned}$$

This means that

$$\begin{aligned} g(2m) &= m(m-1)\cdot\frac{4m-5}{6}+\frac{m(m-1)^2}{2}+\frac{(m-2)(m-1)m}{6} \\ &= \frac{m(m-1)}{6}\cdot((4m-5)+3(m-1)+m-2) \\ &= \frac{m(m-1)}{6}\cdot(8m-10) \\ &= \frac{m(m-1)(4m-5)}{3} \end{aligned}$$

and so $f(2m)=4g(2m)=\dfrac{4m(m-1)(4m-5)}{3}$.
Using this formula,

- when $m=1$, we obtain $f(2)=0$,
- when $m=2$, we obtain $f(4)=\dfrac{4\cdot 2\cdot 1\cdot 3}{3}=8$, and
- when $m=3$, we obtain $f(6)=\dfrac{4\cdot 3\cdot 2\cdot 7}{3}=56$,

which agrees with the specific results that we found earlier.

Thus, $f(40)=\dfrac{4\cdot 20\cdot 19\cdot 75}{3}=38\,000$, which is the answer to (b).
(Note that we could have found this value by calculating the value of $g(40)$ specifically and directly.)

Lastly, we find two even positive integers $n=2m$ with $n<2022$ (that is, with $m<1011$) for which 2022 is a divisor of $f(n)=f(2m)$.
To do this, we note first that $2022=2\cdot 1011=2\cdot 3\cdot 337$ and 337 is a prime number. (How could you verify that 337 is prime?)

Therefore, we need to find two positive integers $m<1011$ for which $\dfrac{4m(m-1)(4m-5)}{3}$ is a multiple of 2, of 3, and of 337.
We can assume that $m\geq 3$, and so we can write m as one of $m=3q$ or $m=3q+1$ or $m=3q+2$ for some positive integer q.
When $m=3q$, we have

$$\frac{4m(m-1)(4m-5)}{3}=\frac{4(3q)(3q-1)(12q-5)}{3}=4q(3q-1)(12q-5)$$

When $m=3q+1$, we have

$$\frac{4m(m-1)(4m-5)}{3}=\frac{4(3q+1)(3q)(12q-1)}{3}=4q(3q+1)(12q-1)$$

When $m=3q+2$, we have

$$\frac{4m(m-1)(4m-5)}{3}=\frac{4(3q+2)(3q+1)(12q+3)}{3}=4(3q+1)(3q+2)(4q+1)$$

Each of these expressions is a positive integer since it is the product of positive integers.
Each of these expressions is even, since each has a factor of 4 in it.
Therefore, we need to determine 2 small enough values of q for which one of these expressions are divisible by 3 and by 337.

Since 337 is a larger prime, we work by looking for factors of 337 first and then checking for factors of 3.

The first two positive multiples of 337 are 337 and 674.
We can check by working through the factors of these three expressions that it is impossible for one of the factors to equal 337 and for another factor in the same expression to be a multiple of 3.
When $q = 224$, we obtain $3q + 2 = 674$ (which is a multiple of 337) and $4q + 1 = 897$ (which is a multiple of 3).
Therefore, when $q = 224$, the expression $4(3q + 1)(3q + 2)(4q + 1)$ is a multiple of 2022, and so setting $m = 3 \cdot 224 + 2 = 674$ tells us that $f(1348)$ is a multiple of 2022.
When $q = 225$, we obtain $3q - 1 = 674$ (which is a multiple of 337) and $q = 225$ (which is a multiple of 3).
Therefore, when $q = 225$, the expression $4q(3q - 1)(12q - 5)$ is a multiple of 2022, and so setting $m = 3 \cdot 225 = 675$ tells us that $f(1350)$ is a multiple of 2022.
This gives us the answer to (c): two even integers n with $n < 2022$ for which $f(n)$ is a multiple of 2022 are $n = 1348$ and $n = 1350$.