The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

cemc.uwaterloo.ca

2019 Canadian Team Mathematics Contest

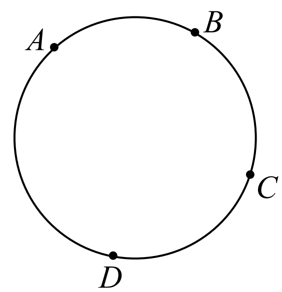
Individual Problems

IMPORTANT NOTES:

- Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) previously stored information such as formulas, programs, notes, etc., (iv) a computer algebra system, (v) dynamic geometry software.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

PROBLEMS:

1. In the diagram, points A, B, C, and D are on a circle. Philippe uses a ruler to connect each pair of these points with a line segment. How many line segments does he draw?



2. What is the smallest positive integer n for which $\sqrt{2019 - n}$ is an integer?
3. At 7:00 a.m. yesterday, Sherry correctly determined what time it had been 100 hours before. What was her answer? (Be sure to include "a.m." or "p.m." in your answer.)
4. A standard die with six faces is tossed onto a table. Itai counts the total number of dots on the five faces that are not lying on the table. What is the probability that this total is at least 19?
5. Suppose that a, b, c, and d are positive integers with $0 < a < b < c < d < 10$. What is the maximum possible value of $\dfrac{a-b}{c-d}$?
6. When the line with equation $y = -2x + 7$ is reflected across the line with equation $x = 3$, the equation of the resulting line is $y = ax + b$. What is the value of $2a + b$?
7. Suppose that $(2^3)^x = 4096$ and that $y = x^3$. What is the ones (units) digit of the integer equal to 3^y?
8. Yasmine makes her own chocolate beverage by mixing volumes of milk and syrup in the ratio $5 : 2$. Milk comes in 2 L bottles and syrup comes in 1.4 L bottles. Yasmine has a limitless supply of full bottles of milk and of syrup. Determine the smallest volume of chocolate beverage that Yasmine can make that uses only whole bottles of both milk and syrup.

9. Suppose that a is an integer. A sequence $x_1, x_2, x_3, x_4, \ldots$ is constructed with

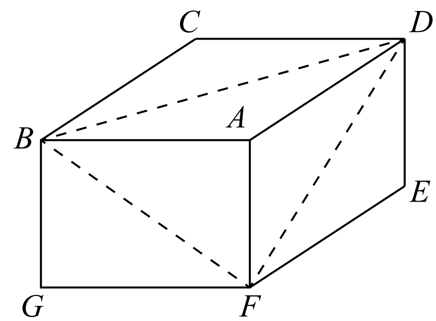
- $x_1 = a$,
- $x_{2k} = 2x_{2k-1}$ for every integer $k \geq 1$, and
- $x_{2k+1} = x_{2k} - 1$ for every integer $k \geq 1$.

For example, if $a = 2$, then

$$x_1 = 2 \qquad x_2 = 2x_1 = 4 \qquad x_3 = x_2 - 1 = 3 \qquad x_4 = 2x_3 = 6 \qquad x_5 = x_4 - 1 = 5$$

and so on. The integer $N = 578$ can appear in this sequence after the 10th term (for example, $x_{12} = 578$ when $a = 10$), but the integer 579 does not appear in the sequence after the 10th term for any value of a. What is the smallest integer $N > 1395$ that could appear in the sequence after the 10th term for some value of a?

10. In the diagram, $ABCDEFGH$ is a rectangular prism. (Vertex H is hidden in this view.) If $\angle ABF = 40°$ and $\angle ADF = 20°$, what is the measure of $\angle BFD$, to the nearest tenth of a degree?



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PROBLEMS:

1. For what value of x is $4x - 8 + 3x = 12 + 5x$?

2. What is the value of $3.5 \times 2.5 + 6.5 \times 2.5$?

3. Ada is younger than Darwyn. Max is younger than Greta. James is older than Darwyn. Max and James are the same age. Which of the five people is the oldest?

4. Determine the average (mean) of $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ as a fraction in lowest terms.

5. Suppose that

$$M = 1^5 + 2^4 \times 3^3 - 4^2 \div 5^1$$
$$N = 1^5 - 2^4 \times 3^3 + 4^2 \div 5^1$$

   What is the value of $M + N$?

6. How many four-digit palindromes $abba$ have the property that the two-digit integer ab and the two-digit integer ba are both prime numbers? (For example, 2332 does not have this property, since 23 is prime but 32 is not.)

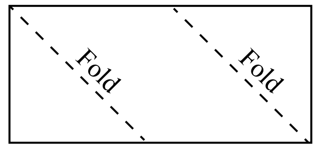
7. Adia writes a list in increasing order of the integers between 1 and 100, inclusive, that cannot be written as the product of two consecutive positive integers. What is the 40th integer in her list?

8. For how many ordered pairs of positive integers (a, b) is $1 < a + b < 22$?

9. Shelly-Ann normally runs along the Laurel Trail at a constant speed of 8 m/s. One day, one-third of the trail is covered in mud, through which Shelly-Ann can only run one-quarter of her normal speed, and it takes her 12 s to run the entire length of the trail. How long is the trail, in metres?

10. Determine the value of a for which $5^a + 5^{a+1} = \sqrt{4500}$.

11. Ezekiel has a rectangular piece of paper with an area of 40. The width of the paper is more than twice the height. He folds the bottom left and top right corners at 45° and creates a parallelogram with an area of 24. What is the perimeter of the original rectangle?



12. What is the value of $123456^2 - 123455 \times 123457$?

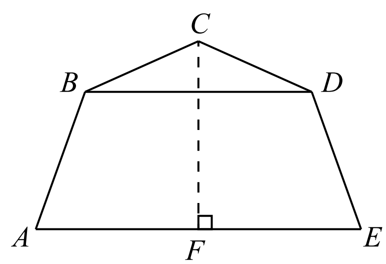
13. Determine the value of $(\log_2 4)(\log_4 6)(\log_6 8)$.

14. The integers x, y and z satisfy $\frac{x}{5} = \frac{6}{y} = \frac{z}{2}$. What is the largest possible value of $x + y + z$?

15. Suppose that $\mathbf{G} = 10^{100}$. ($\mathbf{G}$ is known as a *googol*.) How many times does the digit 9 occur in the integer equal to $\mathbf{G} - 1009^2$?

16. Suppose that $f(x) = x^4 - x^3 - 1$ and $g(x) = x^8 - x^6 - 2x^4 + 1$. If $g(x) = f(x)h(x)$, determine the polynomial function $h(x)$.

17. In the diagram, pentagon $ABCDE$ is symmetrical about altitude CF. Also, $AE = 200$, $CF = 80\sqrt{3}$, $\angle ABC = 150°$, and $\angle BCD = 120°$. Determine the vertical distance between AE and BD.



18. The height of Cylinder A is equal to its diameter. The height and diameter of Cylinder B are each twice those of Cylinder A. The height of Cylinder C is equal to its diameter. The volume of Cylinder C is the sum of the volumes of Cylinders A and B. What is the ratio of the diameter of Cylinder C to the diameter of Cylinder A?

19. Suppose that $f(x) = a(x-b)(x-c)$ is a quadratic function where a, b and c are distinct positive integers less than 10. For each choice of a, b and c, the function $f(x)$ has a minimum value. What is the minimum of these possible minimum values?

20. The integers from 1 to k are concatenated to form the integer $N = 123456789101112\ldots$. Determine the smallest integer value of $k > 2019$ such that N is divisible by 9.

21. The real numbers $x_1, x_2, x_3, \ldots, x_n$ are the consecutive terms of an arithmetic sequence. If

$$\frac{x_2}{x_1 + x_3} + \frac{x_3}{x_2 + x_4} + \frac{x_4}{x_3 + x_5} + \cdots + \frac{x_{n-2}}{x_{n-3} + x_{n-1}} + \frac{x_{n-1}}{x_{n-2} + x_n} = 1957$$

what is the value of n?

22. Suppose that $f_1(x) = \dfrac{1}{2-x}$. For each positive integer $n \geq 2$, define $f_n(x) = f_1(f_{n-1}(x))$ for all real numbers x in the domain of $f_1(f_{n-1}(x))$. The value of $f_{2019}(4)$ can be written as $\dfrac{a}{b}$ where a and b are positive integers with no common divisor larger than 1. What is (a, b)?

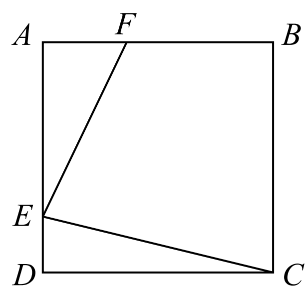
23. The numbers p, q, r, and t satisfy $p < q < r < t$. When these numbers are paired, each pair has a different sum and the four largest sums are 19, 22, 25, and 28. What is the the sum of the possible values for p?

24. Suppose that α and β are the two positive roots of the equation

$$x^2 - \sqrt{13}x^{\log_{13} x} = 0$$

Determine the value of $\alpha\beta$.

25. In the diagram, $ABCD$ is a square. Point F is on AB with $BF = 2AF$. Point E is on AD with $\angle FEC = \angle BCE$. If $0^\circ < \angle ECD < 45^\circ$, what is the value of $\tan(\angle ECD)$?



Relay #1 – Seat 1a

If $x = 1$ and $y = 630$, what is the value of $2019x - 3y - 9$?

Relay #1 – Seat 1b

Let t be TNYWR.

At the start of 2018, the Canadian Excellent Mathematics Corporation had t employees in its Moose Jaw office, 40 employees in its Okotoks office, and no other employees. During 2018, the number of employees in the Moose Jaw office increased by 25% and the number of employees in the Okotoks office decreased by 35%. How many additional employees did the CEMC have at the end of 2018 compared to the beginning of 2018?

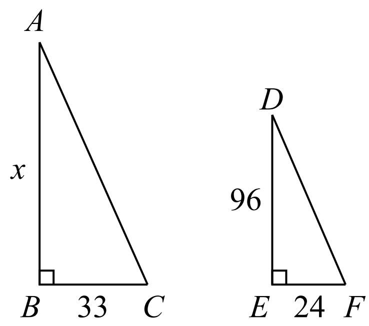
Relay #1 – Seat 1c

Let t be TNYWR.

Kolapo lists the four-digit positive integers that can be made using the digits 2, 4, 5, and 9, each once. Kolapo lists these integers in increasing order. What is the t^{th} number in his list?

Relay #2 – Seat 1a

In the diagram, $\triangle ABC$ is similar to $\triangle DEF$. What is the value of x?



Relay #2 – Seat 1b

Let t be TNYWR.
The sum of the even integers from 2 to $2k$ inclusive equals t for some positive integer k. That is,

$$2 + 4 + 6 + \cdots + (2k - 2) + 2k = t$$

What is the value of k?

Relay #2 – Seat 1c

Let t be TNYWR.
Suppose that O is the origin. Points $P(a, b)$ and $Q(c, 1)$ are in the first quadrant with $a = 2c$.
If the slope of OP is t and the slope of OQ is 1, what is the slope of PQ?

Relay #3 – Seat 1a

How many perfect squares are there between 2 and 150?

Relay #3 – Seat 1b

Let t be TNYWR.
The line with equation $y = -2x+t$ and the parabola with equation $y = (x-1)^2+1$ intersect at point P in the first quadrant. What is the y-coordinate of P?

Relay #3 – Seat 1c

Let t be TNYWR.
The triangle in the first quadrant formed by the x-axis, the y-axis, and the line with equation $(k-1)x+(k+1)y = t$ has area 10. What is the value of k?