The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

cemc.uwaterloo.ca

2018 Canadian Team Mathematics Contest

Individual Problems

IMPORTANT NOTES:

- Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) previously stored information such as formulas, programs, notes, etc., (iv) a computer algebra system, (v) dynamic geometry software.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

PROBLEMS:

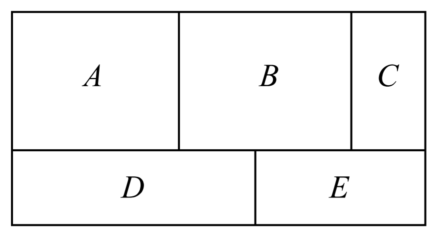
1. The point with coordinates $(a, 0)$ is on the line with equation $y = x + 8$. What is the value of a?

2. If

$$x = \left(1 - \tfrac{1}{12}\right)\left(1 - \tfrac{1}{11}\right)\left(1 - \tfrac{1}{10}\right)\left(1 - \tfrac{1}{9}\right)\left(1 - \tfrac{1}{8}\right)\left(1 - \tfrac{1}{7}\right)\left(1 - \tfrac{1}{6}\right)\left(1 - \tfrac{1}{5}\right)\left(1 - \tfrac{1}{4}\right)\left(1 - \tfrac{1}{3}\right)\left(1 - \tfrac{1}{2}\right)$$

   what is the value of x?

3. In the diagram, a large rectangle is divided into five smaller rectangles which are labelled A, B, C, D, E. In how many ways can exactly two of these five rectangles be shaded so that the shaded rectangles are not touching?



4. The length of the diagonal of a square is 10. What is the area of this square?

5. A three-digit positive integer n has digits abc. (That is, a is the hundreds digit of n, b is the tens digit of n, and c is the ones (units) digit of n.) Determine the largest possible value of n for which

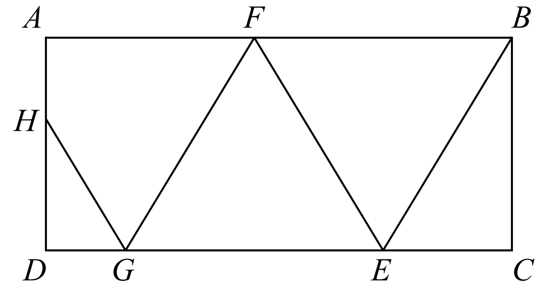
   - a is divisible by 2,
   - the two-digit integer ab (that, a is the tens digit and b is the ones (units) digit) is divisible by 3 but is not divisible by 6, and
   - n is divisible by 5 but is not divisible by 7.

6. Determine all pairs of real numbers (x, y) for which $(4x^2 - y^2)^2 + (7x + 3y - 39)^2 = 0$.

7. An arithmetic sequence has a common difference, d, that is a positive integer and is greater than 1. The sequence includes the three terms 3, 468 and 2018. What is the sum of all of the possible values of d?

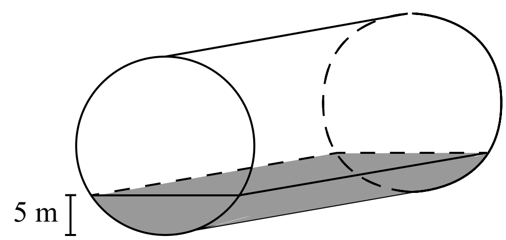
   (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence with common difference 2.)

8. Rectangular room $ABCD$ has mirrors on walls AB and DC. A laser is placed at B. It is aimed at E and the beam reflects off of the mirrors at E, F and G, arriving at H. The laws of physics tell us that $\angle BEC = \angle FEG$ and $\angle BFE = \angle AFG$ and $\angle FGE = \angle HGD$. If $AB = 18$ m, $BC = 10$ m and $HD = 6$ m, what is the total length of the path $BEFGH$ travelled by the laser beam?



9. A box contains R red balls, B blue balls, and no other balls. One ball is removed and set aside, and then a second ball is removed. On each draw, each ball in the box is equally likely to be removed. The probability that both of these balls are red is $\frac{2}{7}$. The probability that exactly one of these balls is red is $\frac{1}{2}$. Determine the pair (R, B).

10. A cylindrical tank has radius 10 m and length 30 m. The tank is lying on its side on a flat surface and is filled with water to a depth of 5 m. Water is added to the tank and the depth of the water increases from 5 m to $10 + 5\sqrt{2}$ m. If the volume of water added to the tank, in m^3, can be written as $a\pi + b + c\sqrt{p}$ for some integers a, b, c and prime number p, determine the quadruple (a, b, c, p).



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Team Problems

IMPORTANT NOTES:

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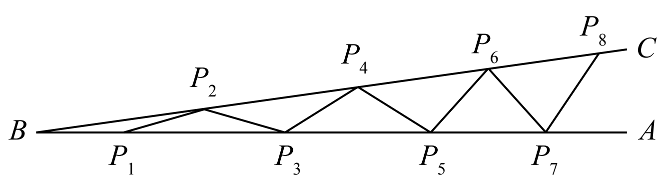
PROBLEMS:

1. What is the value of $\sqrt{1+2+3+4+5+6+7+8}$?
2. A bucket that is $\frac{2}{3}$ full contains 9 L of maple syrup. What is the capacity of the bucket, in litres?
3. The sum of four consecutive odd integers is 200. What is the largest of these four integers?
4. It takes 18 doodads to make 5 widgets. It takes 11 widgets to make 4 thingamabobs. How many doodads does it take to make 80 thingamabobs?
5. In the diagram, points P_1, P_3, P_5, P_7 are on BA and points P_2, P_4, P_6, P_8 are on BC such that $BP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = P_5P_6 = P_6P_7 = P_7P_8$. If $\angle ABC = 5°$, what is the measure of $\angle AP_7P_8$?



6. A *polygonal pyramid* is a three-dimensional solid. Its base is a regular polygon. Each of the vertices of the polygonal base is connected to a single point, called the *apex*. The sum of the number of edges and the number of vertices of a particular polygonal pyramid is 1915. How many faces does this pyramid have?
7. There are four ways to evaluate the expression "$\pm 2 \pm 5$":

$$2+5=7 \qquad 2-5=-3 \qquad -2+5=3 \qquad -2-5=-7$$

There are eight ways to evaluate the expression "$\pm 2^{11} \pm 2^5 \pm 2$". When these eight values are listed in decreasing order, what is the third value in the list?

8. For how many positive integers n is the sum

$$(-n)^3 + (-n+1)^3 + \cdots + (n-2)^3 + (n-1)^3 + n^3 + (n+1)^3$$

less than 3129?

9. For real numbers a and b, we define $a \nabla b = ab - ba^2$. For example, $5 \nabla 4 = 5(4) - 4(5^2) = -80$. Determine the sum of the values of x for which $(2 \nabla x) - 8 = x \nabla 6$.

10. Birgit has a list of four numbers. Luciano adds these numbers together, three at a time, and gets the sums 415, 442, 396, and 325. What is the sum of Birgit's four numbers?

11. On Monday, Krikor left his house at 8:00 a.m., drove at a constant speed, and arrived at work at 8:30 a.m. On Tuesday, he left his house at 8:05 a.m., drove at a constant speed, and arrived at work at 8:30 a.m. By what percentage did he increase his speed from Monday to Tuesday?

12. What is the value of $\pi \log_{2018} \sqrt{2} + \sqrt{2} \log_{2018} \pi + \pi \log_{2018} \left(\frac{1}{\sqrt{2}}\right) + \sqrt{2} \log_{2018} \left(\frac{1}{\pi}\right)$?

13. An Eilitnip number is a three-digit positive integer with the properties that, for some integer k with $0 \leq k \leq 7$:

    - its digits are k, $k+1$ and $k+2$ in some order, and
    - it is divisible by $k+3$.

    Determine the number of Eilitnip numbers.

14. An arithmetic sequence has 2036 terms labelled $t_1, t_2, t_3, \ldots, t_{2035}, t_{2036}$. Its 2018th term is $t_{2018} = 100$. Determine the value of $t_{2000} + 5t_{2015} + 5t_{2021} + t_{2036}$.

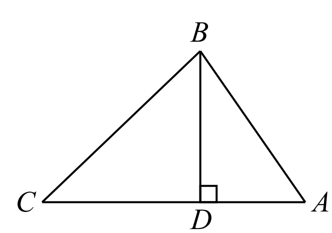
    (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence with common difference 2.)

15. A square wall has side length n metres. Guillaume paints n non-overlapping circular targets on the wall, each with radius 1 metre. Mathilde is going to throw a dart at the wall. Her aim is good enough to hit the wall at a single point, but poor enough that the dart will hit a random point on the wall. What is the largest possible value of n so that the probability that Mathilde's dart hits a target is at least $\frac{1}{2}$?

16. Determine the largest positive integer n for which 7^n is a divisor of the integer $\dfrac{200!}{90!30!}$.

    (Note: If n is a positive integer, the symbol $n!$ (read "n factorial") is used to represent the product of the integers from 1 to n. That is, $n! = n(n-1)(n-2)\cdots(3)(2)(1)$. For example, $5! = 5(4)(3)(2)(1)$ or $5! = 120$.)

17. In the diagram, D is on side AC of $\triangle ABC$ so that BD is perpendicular to AC. Also, $\angle BAC = 60^\circ$ and $\angle BCA = 45^\circ$. If the area of $\triangle ABC$ is $72 + 72\sqrt{3}$, what is the length of BD?



18. BBAAABA and AAAAAAA are examples of "words" that are seven letters long where each letter is either A or B. How many seven-letter words in which each letter is either A or B do not contain three or more A's in a row?

19. Determine all real numbers x for which $x^{3/5} - 4 = 32 - x^{2/5}$.

20. In $\triangle ABC$, $BC = AC - 1$ and $AC = AB - 1$. If $\cos(\angle BAC) = \frac{3}{5}$, determine the perimeter of $\triangle ABC$.

21. The function f has the following properties:

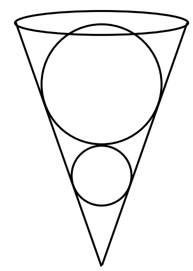
    (i) its domain is all real numbers,

    (ii) it is an odd function (that is, $f(-x) = -f(x)$ for every real number x), and

    (iii) $f(2x-3) - 2f(3x-10) + f(x-3) = 28 - 6x$ for every real number x.

    Determine the value of $f(4)$.

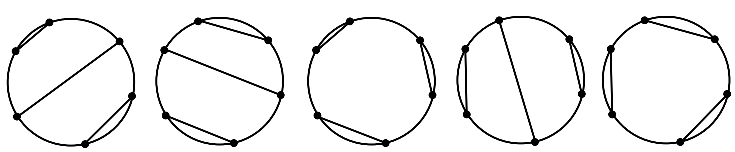
22. A right circular cone contains two spheres, as shown. The radius of the larger sphere is 2 times the radius of the smaller sphere. Each sphere is tangent to the other sphere and to the lateral surface of the cone. The larger sphere is tangent to the cone's circular base. Determine the fraction of the cone's volume that is *not* occupied by the two spheres.



23. Let a be a fixed real number. Define $M(t)$ to be the maximum value of $-x^2 + 2ax + a$ over all real numbers x with $x \leq t$. Determine a polynomial expression in terms of a that is equal to $M(a-1) + M(a+2)$ for every real number a.

24. The graphs $y = 2\cos 3x + 1$ and $y = -\cos 2x$ intersect at many points. Two of these points, P and Q, have x-coordinates between $\frac{17\pi}{4}$ and $\frac{21\pi}{4}$. The line through P and Q intersects the x-axis at B and the y-axis at A. If O is the origin, what is the area of $\triangle BOA$?

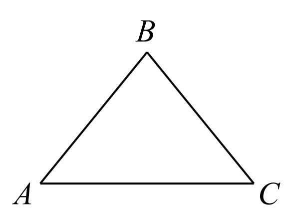
25. There are 16 distinct points on a circle. Determine the number of different ways to draw 8 non-intersecting line segments connecting pairs of points so that each of the 16 points is connected to exactly one other point. (For example, when the number of points is 6, there are 5 different ways to connect them, as shown.)



0 (a). Evaluate $\dfrac{9+3\times 3}{3}$.

0 (b). Let t be TNYWR.
What is the area of a triangle with base $2t$ and height $3t+2$?

0 (c). Let t be TNYWR.
In the diagram, $\triangle ABC$ is isosceles with $AB = BC$. If $\angle ABC = t^\circ$, what is the measure of $\angle BAC$, in degrees?



1 (a). The integers 390 and 9450 have three common positive divisors that are prime numbers. What is the sum of these prime numbers?

1 (b). Let t be TNYWR.

If $n = \dfrac{(4t^2 - 10t - 2) - 3(t^2 - t + 3) + (t^2 + 5t - 1)}{(t + 7) + (t - 13)}$, what is the value of n?

1 (c). Let t be TNYWR.

Azmi has two fair dice, each with six sides.

The sides of one of the dice are labelled $1, 2, 3, 4, 5, 6$.

The sides of the other die are labelled $t - 10, t, t + 10, t + 20, t + 30, t + 40$.

When these two dice are rolled, there are 36 different possible values for the sum of the numbers on the top faces. What is the average of these 36 possible sums?

2 (a). The expression $2(x-3)^2 - 12$ can be re-written as $ax^2 + bx + c$ for some numbers a, b, c. What is the value of $10a - b - 4c$?

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2 (b). Let t be TNYWR.
The line ℓ passes through the points $(-4, t)$ and (k, k) for some real number k.
The line ℓ is perpendicular to the line passing through the points $(11, -7)$ and $(15, 5)$.
What is the value of k?

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2 (c). Let t be TNYWR.
In a magic square, the sum of the numbers in each column, the sum of the numbers in each row, and the sum of the numbers on each diagonal are all the same. In the magic square shown, what is the value of N?

	$3t-2$	$4t-6$	
$4t-1$	$2t+12$	$t+16$	$3t+1$
N	$4t-2$		$t+15$
			$4t-5$

3 (a). The line with equation $y = 5x + a$ passes through the point (a, a^2). If $a \neq 0$, what is the value of a?

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3 (b). Let t be TNYWR.
The CEMC Compasses basketball team scored exactly $10t$ points in each of 4 games and scored exactly 20 points in each of g games. Over this set of games, they scored an average of 28 points per game. What is the value of g?

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3 (c). Let t be TNYWR.
The pair $(x, y) = (a, b)$ is a solution of the system of equations

$$x^2 + 4y = t^2$$
$$x^2 - y^2 = 4$$

If $b > 0$, what is the value of b?