The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2014 Canadian Team Mathematics Contest

Individual Problems

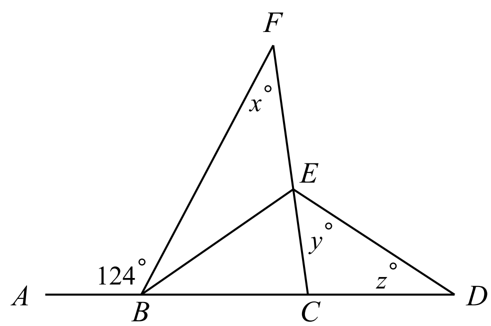
1. The sum of 8 one-digit positive integers is 68. If seven of these integers are equal, determine the other integer.

2. The five-digit positive integer $15AB9$ is a perfect square for some digits A and B. What is the value of $A + B$?

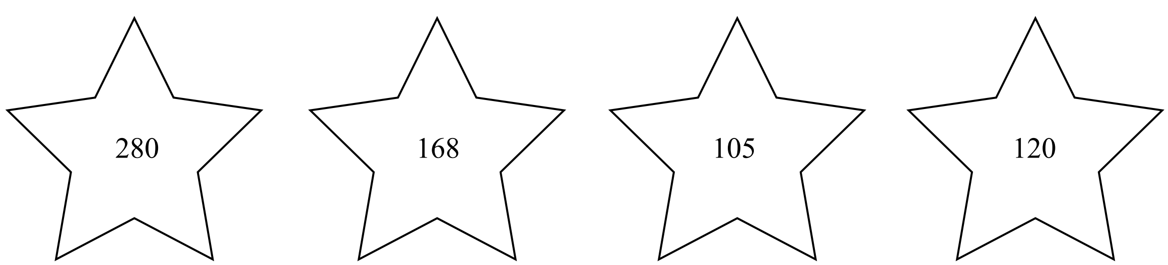
3. If $b \neq 0$, we define $a \heartsuit b = ab - \frac{a}{b}$. For example, $3 \heartsuit 1 = 3(1) - \frac{3}{1} = 0$.
If $x = 4 \heartsuit 2$ and $y = 2 \heartsuit 2$, what is the value of x^y ?

4. The numbers $36, 27, 42, 32, 28, 31, 23, 17$ are grouped in pairs so that the sum of each pair is the same. Which number is paired with 32?

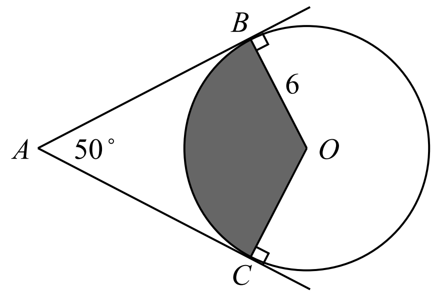
5. In the diagram, points B and C lie on AD and point E lies on CF. The measures of four angles are shown. What is the value of $x + y + z$?



6. In the diagram, a positive integer is hidden behind each star. The integer shown on each star is the product of the integers hidden behind the other three stars. What is the product of all four hidden integers?



7. In the diagram, the circle has centre O and radius 6. Point A is outside the circle and points B and C are on the circle so that AB is perpendicular to BO, AC is perpendicular to CO, and $\angle BAC = 50^\circ$. What is the area of the shaded region?



8. When three students, Al, Betty, and Charlie, compete in a competition, there are 13 possible orders of finish, allowing for the possibility of ties. These possibilities are illustrated in the chart below:

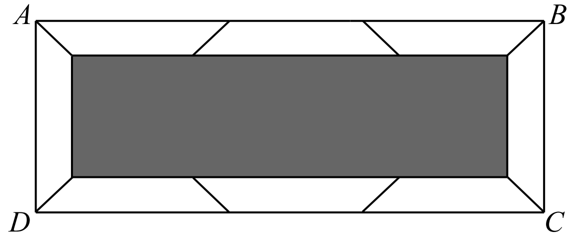
First	A	A	B	B	C	C	ABC	AB	AC	BC	A	B	C
Second	B	C	A	C	A	B		C	B	A	BC	AC	AB
Third	C	B	C	A	B	A							

When four students, David, Erin, Frank, and Greg, compete in a competition, how many possible orders of finish are there, allowing for the possibility of ties?

9. In a geometric sequence with five terms t_1, t_2, t_3, t_4, t_5, each term is positive and $t_1 > t_2$. If the sum of the first two terms is $\frac{15}{2}$ and the sum of the squares of the first two terms is $\frac{153}{4}$, what is the value of t_5?

(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

10. The diagram shows a rectangular picture frame $ABCD$ made of eight identical trapezoids. The shaded region is where the picture goes. The length, AB, and width, AD, of the frame are both positive integers. The area of each individual trapezoidal piece is a prime number. If the area of the shaded region is less than 2000 square units, what is the maximum possible area of the shaded region?



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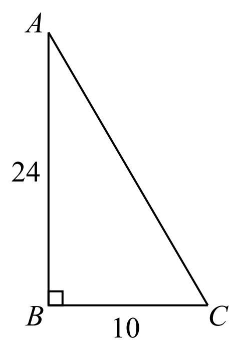
Team Problems

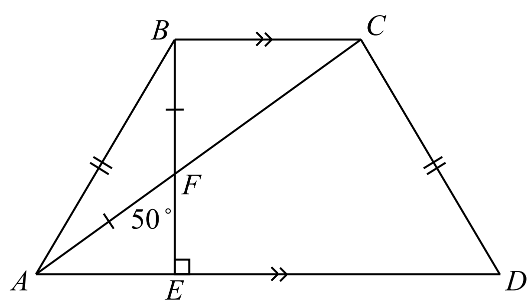
1. Beginning at 100, David counts down by 11 at a time. What is the first number less than 0 that David will count?

2. What is the value of $1 + 2 + 3 + \cdots + 18 + 19 + 20$? (That is, what is the sum of the first 20 positive integers?)

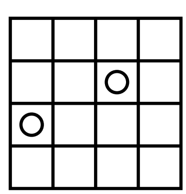
3. In the diagram, $\triangle ABC$ is right-angled at B with $AB = 24$ and $BC = 10$. If AB and BC are each increased by 6, by how much does AC increase?



4. In the diagram, $ABCD$ is a trapezoid with BC parallel to AD and $AB = CD$. Point E is on AD so that BE is perpendicular to AD and point F is the point of intersection of AC with BE. If $AF = FB$ and $\angle AFE = 50^\circ$, what is the measure of $\angle ADC$?



5. In the diagram below, how many rectangles contain both circles?



6. What is the smallest positive integer that is a multiple of each of 2, 4, 6, 8, 10, 12, 14, 16, 18, and 20?

7. Integers x and y, with $x > y$, satisfy $x + y = 7$ and $xy = 12$.
Integers m and n, with $m > n$, satisty $m + n = 13$ and $m^2 + n^2 = 97$.
If $A = x - y$ and $B = m - n$, determine the value of $A - B$.

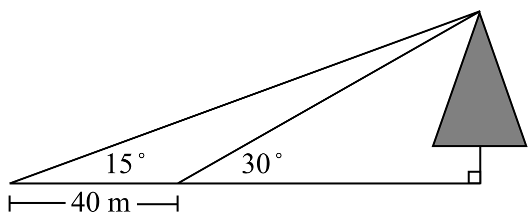
8. Robert was born in the year n^2. On his birthday in the year $(n + 1)^2$, he will be 89 years old. In what year was he born?

9. Suppose that

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)=2014$$

for some positive integer k. (There are k factors in the product.) What is the value of k?

10. Rowena has a very long, level backyard. From a particular point, she determines that the angle of elevation to a large tree in her backyard is 15°. She moves 40 m closer and determines that the new angle of elevation is 30°. How tall, in metres, is the tree?



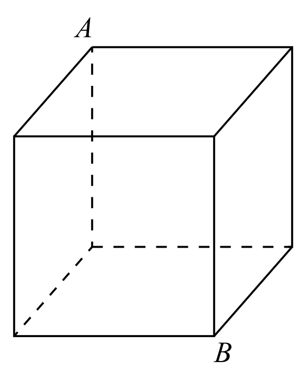
11. Line segment PQ has endpoints at $P(6, -2)$ and $Q(-3, 10)$. Point $R(a, b)$ is a point on PQ such that the distance from P to R is one-third the distance from P to Q. What is $b - a$?

12. In how many ways can 105 be expressed as the sum of at least two consecutive positive integers?

13. An insect travels from vertex A to vertex B on the cube shown with edges of length 1 metre. The insect can travel in exactly one and only one of the following ways:

- crawl along the edges of the cube (it crawls at 5 metres per minute along edges)
- crawl along the faces of the cube (it crawls at 4 metres per minute along faces)
- fly through the interior of the cube (it flies at 3 metres per minute along any path through the interior of the cube)

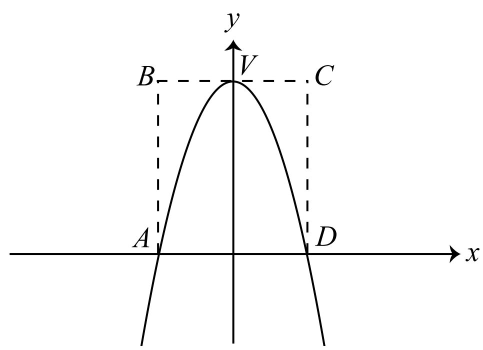
What is the shortest possible time, in minutes, that the insect can take to get from A to B?



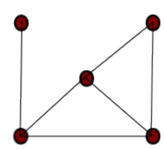
14. In an arithmetic sequence with 20 terms, the sum of the first three terms is 15 and the sum of the last three terms is 12. What is the sum of the 20 terms in the sequence?

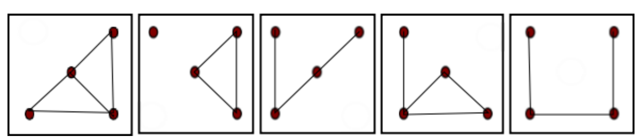
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.)

15. A parabola has equation $y = k^2 - x^2$, for some positive number k. Rectangle $ABCD$ is drawn with sides parallel to the axes so that A and D are the points where the parabola intersects the x-axis and so that the vertex, V, of the parabola is the midpoint of BC. If the perimeter of the rectangle is 48, what is the value of k?

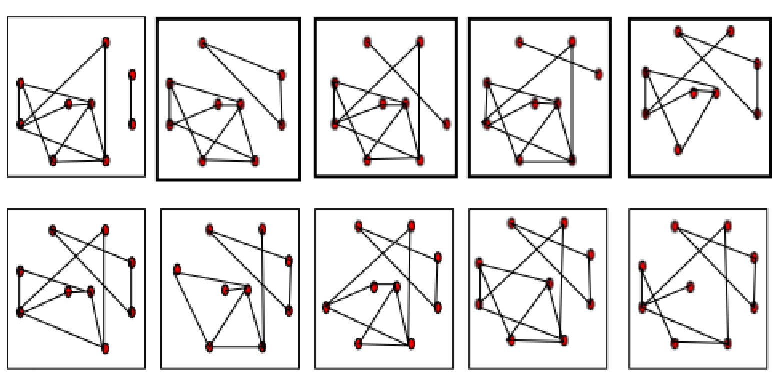


16. A *graph* is a diagram consisting of points (called *vertices*) and line segments joining some pairs of points (called *edges*).
Starting with an original graph with five vertices, as shown below, five new graphs are formed by removing one of the vertices from the original graph and all of the edges joined to this vertex.





This process is repeated starting with an original graph with 10 vertices.



How many edges did this second original graph have?

17. A quantity of grey paint has a mass of 12 kg. The grey paint is a mixture of black paint and white paint, of which 80% by mass is white paint. More white paint is added until the mixture is 90% white paint by mass. What is the resulting total mass of the paint, in kg?

18. In a regular polygon, a diagonal is a line segment joining a vertex to any non-neighbouring vertex. For example, a regular hexagon has 9 diagonals. If a regular polygon with n sides has 90 diagonals, what is the value of n?

19. Determine the number of triples (a, b, c) of three positive integers with $a < b < c$ whose sum is 100 and whose product is 18 018.

20. We define $(a, b) \diamond (c, d) = (ac - bd, ad + bc)$.
Determine all pairs (x, y) for which $(x, 3) \diamond (x, y) = (6, 0)$.

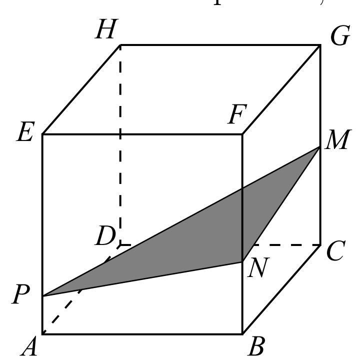
21. A *standard die* is a cube whose faces are labelled 1, 2, 3, 4, 5, 6. Raffi rolls a standard die three times obtaining the results p, q and r on the first, second and third rolls, respectively. What is the probability that $p < q < r$?

22. Determine the largest possible radius of a circle that is tangent to both the x-axis and y-axis, and passes through the point $(9, 2)$.

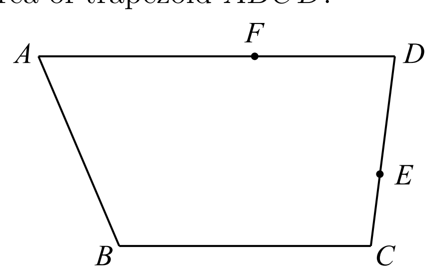
(A circle in the xy-plane with centre (a, b) and radius r has equation $(x - a)^2 + (y - b)^2 = r^2$.)

23. For each positive integer n, define $s(n)$ to be the sum of the digits of n. For example, $s(2014) = 2 + 0 + 1 + 4 = 7$. Determine all positive integers n with $1000 \le n \le 9999$ for which $\dfrac{n}{s(n)} = 112$.

24. In the diagram, cube $ABCDEFGH$ has edges of length a. Points M, N and P are on edges GC, FB, and EA, respectively, so that $MC = \frac{1}{2}a$, $NB = \frac{1}{3}a$ and $PA = \frac{1}{4}a$. If the plane defined by points M, N and P intersects HD at point T, determine the length DT.



25. In trapezoid $ABCD$, AD is parallel to BC and $BC : AD = 5 : 7$. Point F lies on AD and point E lies on DC so that $AF : FD = 4 : 3$ and $CE : ED = 2 : 3$. If the area of quadrilateral $ABEF$ is 123, determine the area of trapezoid $ABCD$.



0 (a). Evaluate $2 \times 0 + 1 \times 4$.

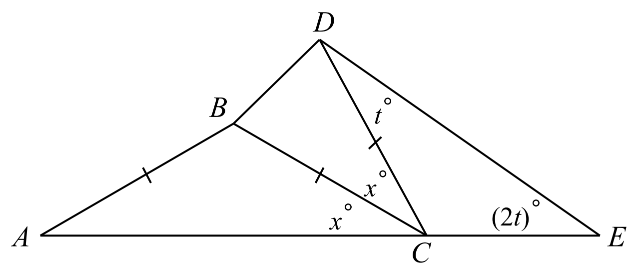
0 (b). Let t be TNYWR.
The average of the list of five numbers $13, 16, 10, 15, 11$ is m.
The average of the list of four numbers $16, t, 3, 13$ is n.
What is the value of $m - n$?

0 (c). Let t be TNYWR.
The lines with equations $y = 12$ and $y = 2x + t$ intersect at the point (a, b). What is the value of a?

1 (a). Evaluate $\frac{1}{2}\left(\frac{1}{\frac{1}{9}}+\frac{1}{\frac{1}{6}}-\frac{1}{\frac{1}{5}}\right)$.

1 (b). Let t be TNYWR.
Determine the positive integer x that satisfies $2 : m : t = m : 32 : x$.

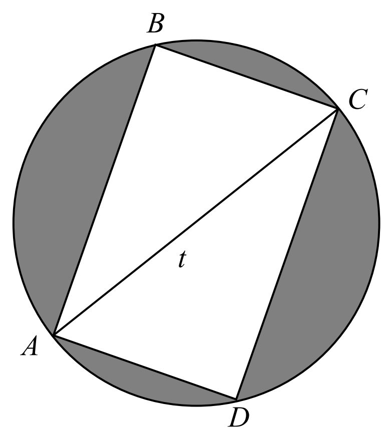
1 (c). Let t be TNYWR.
In the diagram, C lies on AE and $AB = BC = CD$. If $\angle CDE = t^\circ$, $\angle DEC = (2t)^\circ$, and $\angle BCA = \angle BCD = x^\circ$, determine the measure of $\angle ABC$.



2 (a). Suppose that a and b are positive integers with $2^a \times 3^b = 324$.
Evaluate $2^b \times 3^a$.

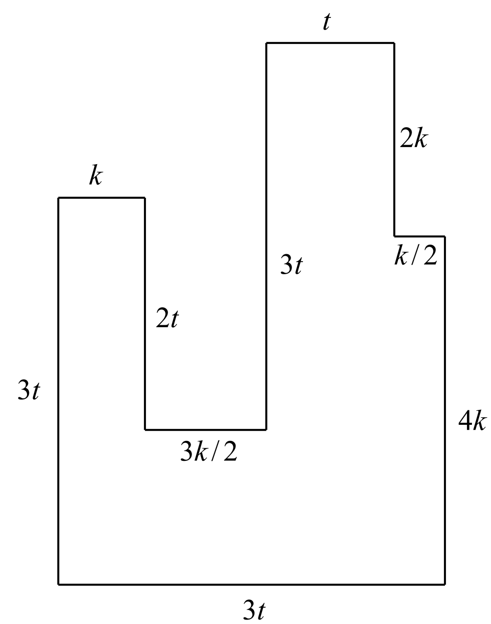
2 (b). Let t be TNYWR.
Three siblings share a box of chocolates that contains t pieces. Sarah eats $\frac{1}{3}$ of the total number of chocolates and Andrew eats $\frac{3}{8}$ of the total number of chocolates. Cecily eats the remaining chocolates in the box. How many more chocolates does Sarah eat than Cecily eats?

2 (c). Let t be TNYWR.
In the diagram, the vertices of rectangle $ABCD$ lie on a circle. Diagonal AC is a diameter of the circle and has length t. If $CD = 2AD$, find the area of the shaded region, and write your answer in the form $a\pi - \dfrac{b}{c}$ with a, b, c positive integers and with b and c having no common positive divisor larger than 1.



3 (a). What is the greatest common divisor of the three integers 36, 45 and 495?

3 (b). Let t be TNYWR.
In the diagram, all line segments meet at right angles. If the perimeter of the given shape is 162 units, what is the value of k?



3 (c). Let t be TNYWR.
The expression $(tx + 3)^3$ can be re-written in the form $ax^3 + bx^2 + cx + d$ for some positive integers a, b, c, d. Determine the value of the largest of a, b, c, and d.