The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Euclid Contest

Wednesday, April 17, 2013
(in North America and South America)

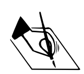
Thursday, April 18, 2013
(outside of North America and South America)

Do not open this booklet until instructed to do so.

Time: $2\frac{1}{2}$ hours
Calculators are permitted, provided they are non-programmable and without graphic displays.

Number of questions: 10
Each question is worth 10 marks

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by 
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by 
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express calculations and answers as exact numbers such as $\pi + 1$ and $\sqrt{2}$, etc., rather than as $4.14\ldots$ or $1.41\ldots$, except where otherwise indicated.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, http://www.cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

TIPS:	1. Please read the instructions on the front cover of this booklet. 2. Write all answers in the answer booklet provided. 3. For questions marked, place your answer in the appropriate box in the answer booklet and **show your work**. 4. For questions marked, provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution. 5. Diagrams are *not* drawn to scale. They are intended as aids only.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth, grade, and sex, on the Student Information Form, and that you have answered the question about eligibility.

1. (a) What is the smallest positive integer x for which $\sqrt{113+x}$ is an integer?

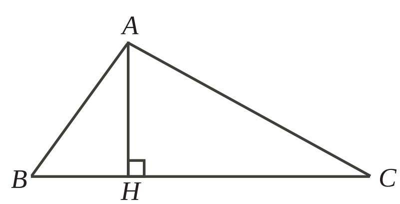
   (b) The average of 3 and 11 is a. The average of a and b is 11. What is the value of b?

   (c) Charlie is 30 years older than his daughter Bella. Charlie is also six times as old as Bella. Determine Charlie's age.

2. (a) If $\dfrac{21}{x} = \dfrac{7}{y}$ with $x \neq 0$ and $y \neq 0$, what is the value of $\dfrac{x}{y}$?

   (b) For which positive integer n are both $\dfrac{1}{n+1} < 0.2013$ and $0.2013 < \dfrac{1}{n}$ true?

   (c) In the diagram, H is on side BC of $\triangle ABC$ so that AH is perpendicular to BC. Also, $AB = 10$, $AH = 8$, and the area of $\triangle ABC$ is 84. Determine the perimeter of $\triangle ABC$.



3. (a) In the Fibonacci sequence, $1, 1, 2, 3, 5, \ldots$, each term after the second is the sum of the previous two terms. How many of the first 100 terms of the Fibonacci sequence are odd?

   (b) In an arithmetic sequence, the sum of the first and third terms is 6 and the sum of the second and fourth terms is 20. Determine the tenth term in the sequence.

   (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.)

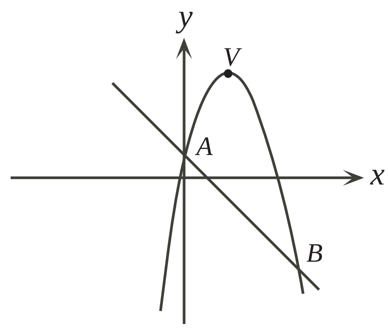
4. (a) How many positive integers less than 1000 have only odd digits?

(b) Determine all ordered pairs (a, b) that satisfy the following system of equations.

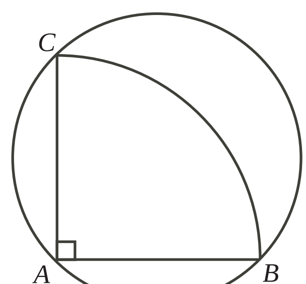
$$\begin{aligned} a+b &= 16 \\ \frac{4}{7} &= \frac{1}{a}+\frac{1}{b} \end{aligned}$$

5. (a) Tanner has two identical dice. Each die has six faces which are numbered 2, 3, 5, 7, 11, 13. When Tanner rolls the two dice, what is the probability that the sum of the numbers on the top faces is a prime number?

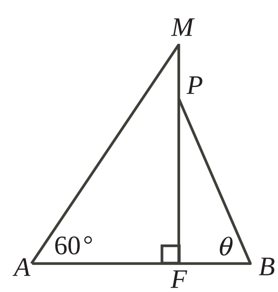
(b) In the diagram, V is the vertex of the parabola with equation $y = -x^2 + 4x + 1$. Also, A and B are the points of intersection of the parabola and the line with equation $y = -x + 1$. Determine the value of $AV^2 + BV^2 - AB^2$.



6. (a) In the diagram, ABC is a quarter of a circular pizza with centre A and radius 20 cm. The piece of pizza is placed on a circular pan with A, B and C touching the circumference of the pan, as shown. What fraction of the pan is covered by the piece of pizza?



(b) The deck AB of a sailboat is 8 m long. Rope extends at an angle of $60°$ from A to the top (M) of the mast of the boat. More rope extends at an angle of θ from B to a point P that is 2 m below M, as shown. Determine the height MF of the mast, in terms of θ.



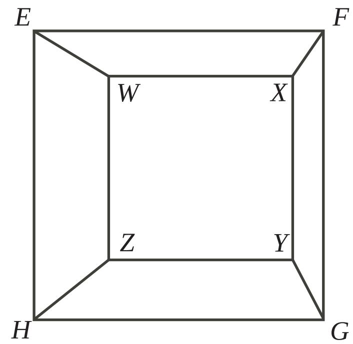
7. (a) If $\dfrac{1}{\cos x} - \tan x = 3$, what is the numerical value of $\sin x$?

(b) Determine all linear functions $f(x) = ax + b$ such that if $g(x) = f^{-1}(x)$ for all values of x, then $f(x) - g(x) = 44$ for all values of x. (Note: f^{-1} is the inverse function of f.)

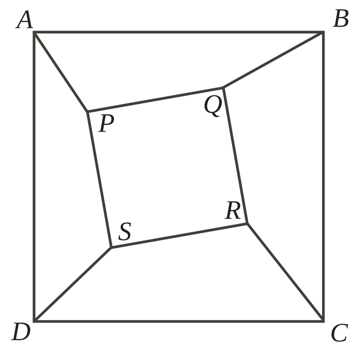
8\. (a) Determine all pairs (a, b) of positive integers for which $a^3 + 2ab = 2013$.

(b) Determine all real values of x for which $\log_2(2^{x-1} + 3^{x+1}) = 2x - \log_2(3^x)$.

9\. (a) Square $WXYZ$ has side length 6 and is drawn, as shown, completely inside a larger square $EFGH$ with side length 10, so that the squares do not touch and so that WX is parallel to EF. Prove that the sum of the areas of trapezoid $EFXW$ and trapezoid $GHZY$ does not depend on the position of $WXYZ$ inside $EFGH$.



(b) A large square $ABCD$ is drawn, with a second smaller square $PQRS$ completely inside it so that the squares do not touch. Line segments AP, BQ, CR, and DS are drawn, dividing the region between the squares into four non-overlapping convex quadrilaterals, as shown. If the sides of $PQRS$ are *not* parallel to the sides of $ABCD$, prove that the sum of the areas of quadrilaterals $APSD$ and $BCRQ$ equals the sum of the areas of quadrilaterals $ABQP$ and $CDSR$. (Note: A convex quadrilateral is a quadrilateral in which the measure of each of the four interior angles is less than 180°.)



10\. A *multiplicative partition* of a positive integer $n \geq 2$ is a way of writing n as a product of one or more integers, each greater than 1. Note that we consider a positive integer to be a multiplicative partition of itself. Also, the order of the factors in a partition does not matter; for example, $2 \times 3 \times 5$ and $2 \times 5 \times 3$ are considered to be the same partition of 30. For each positive integer $n \geq 2$, define $P(n)$ to be the number of multiplicative partitions of n. We also define $P(1) = 1$. Note that $P(40) = 7$, since the multiplicative partitions of 40 are 40, 2×20, 4×10, 5×8, $2 \times 2 \times 10$, $2 \times 4 \times 5$, and $2 \times 2 \times 2 \times 5$.

(a) Determine the value of $P(64)$.

(b) Determine the value of $P(1000)$.

(c) Determine, with proof, a sequence of integers $a_0, a_1, a_2, a_3, \ldots$ with the property that

$$P(4 \times 5^m) = a_0 P(2^m) + a_1 P(2^{m-1}) + a_2 P(2^{m-2}) + \cdots + a_{m-1} P(2^1) + a_m P(2^0)$$

for every positive integer m.

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For students...

Thank you for writing the 2013 Euclid Contest!
In 2012, more than 16 000 students from around the world registered to write the Euclid Contest.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2013 Canadian Senior Mathematics Contest, which will be written in November 2013.

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