The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

2011 Fryer Contest

Wednesday, April 13, 2011

Solutions

1. (a) The 5^{th} term of the sequence is 14 and the common difference is 3.
Therefore, the 6^{th} term of the sequence is $14+3$ or 17, and the 7^{th} term of the sequence is $17+3$ or 20.

(b) Each term after the 1^{st} is 3 more than the term to its left.
To get from the 1^{st} term to the 31^{st} term, we move 30 times to the right.
Therefore, the 31^{st} term is 30×3 more than the 1^{st} term, or $2+30(3)=92$.

(c) Using our work from part (b), we want to determine how many times 3 must be added to the first term 2 so that the result is 110.
That is, how many 3s give $110-2$ or 108.
Since $108 \div 3 = 36$, then 36 3s must be added to 2 to give 110.
Therefore if the last term is 110, then the number of terms in the sequence is $36+1$ or 37.

(d) If 1321 appears in the sequence, then some integer number of 3s added to the first term 2 is equal to 1321.
That is, we want to determine how many 3s give $1321-2$ or 1319.
Since 3 does not divide 1319 ($\frac{1319}{3} = 439\frac{2}{3}$), there is no integer number of 3s that when added to the first term 2 is equal to 1321.
Thus, 1321 does not appear in the sequence.

2. (a) (i) Since $AB = AC$, then $\triangle ABC$ is isosceles.
Therefore, altitude AD bisects the base BC so that $BD = DC = \frac{14}{2} = 7$.
Since $\angle ADB = 90^\circ$, $\triangle ADB$ is right angled.
By the Pythagorean Theorem, $25^2 = AD^2 + 7^2$ or $AD^2 = 25^2 - 7^2$ or $AD^2 = 625 - 49 = 576$, and so $AD = \sqrt{576} = 24$, since $AD > 0$.

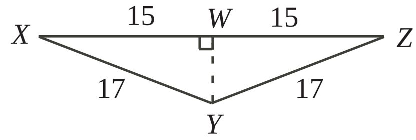
(ii) The area of $\triangle ABC$ is $\frac{1}{2} \times BC \times AD$ or $\frac{1}{2} \times 14 \times 24 = 168$.

(b) (i) Through the process described, $\triangle ADB$ is rotated 90° counter-clockwise about D to become $\triangle PDQ$.
Similarly, $\triangle ADC$ is rotated 90° clockwise about D to become $\triangle RDQ$.
Through both rotations, the lengths of the sides of the original triangles remain unchanged.
Thus, $PD = AD = 24$ and $RD = AD = 24$.
Since P, D and R lie in a straight line, then base $PR = PD + RD = 24 + 24 = 48$.

(ii) *Solution 1*
When $\triangle ADC$ is rotated 90° clockwise about D, side DC becomes altitude DQ in $\triangle PQR$.
Therefore, $DQ = DC = 7$.
Thus, the area of $\triangle PQR$ is $\frac{1}{2} \times PR \times DQ$ or $\frac{1}{2} \times 48 \times 7 = 168$.

Solution 2
As determined in part (a), the area of $\triangle ABC = 168$.
The area of $\triangle ABC$ is equal to the area of $\triangle ADB$ added to the area of $\triangle ADC$.
Thus, the sum of the areas of these two smaller triangles is also 168.
Through the rotations described, triangles ADB and ADC remain unchanged.
Since $\triangle PDQ$ is congruent to $\triangle ADB$, their areas are equal.
Since $\triangle RDQ$ is congruent to $\triangle ADC$, their areas are equal.
Therefore, the area of $\triangle PQR$ is equal to the sum of the areas of triangles PDQ and RDQ, which equals the sum of the areas of triangles ADB and ADC or 168.

(c) Since $XY = YZ$, then $\triangle XYZ$ is isosceles.
Draw altitude YW from Y to W on XZ.
Altitude YW bisects the base XZ so that $XW = WZ = \frac{30}{2} = 15$, as shown.



Since $\angle YWX = 90^\circ$, $\triangle YWX$ is right angled.
By the Pythagorean Theorem, $17^2 = YW^2 + 15^2$ or $YW^2 = 17^2 - 15^2$ or $YW^2 = 289 - 225 = 64$, and so $YW = \sqrt{64} = 8$, since $YW > 0$.
By reversing the process described in part (b), we rotate $\triangle XWY$ clockwise 90° about W and similarly rotate $\triangle ZWY$ counter-clockwise 90° about W to obtain a new isosceles triangle with the same area.
The new triangle formed has two equal sides of length 17 (since XY and ZY form these sides) and a third side having length twice that of YW or $2 \times 8 = 16$ (since the new base consists of two copies of YW).

3. (a)

Two-digit number	Step 1	Step 2	Step 3
68	$6 \times 8 = 48$	$4 \times 8 = 32$	$3 \times 2 = 6$

Beginning with 68, the process stops after 3 steps.

(b) For the process to stop at 8, the preceding number must have two digits whose product is 8.
The factor pairs of 8 are: 1×8; 2×4.
Thus, the only two-digit numbers whose digits have a product of 8 are 18, 81, 24, and 42.
Next, consider the factor pairs of 18, 81, 24, and 42.
The factor pairs of 18 are: 1×18; 2×9; 3×6.
Since the factors 1 and 18 cannot be used to form a two-digit number, the only two-digit numbers whose digits have a product of 18 are $29, 92, 36, 63$.
The factor pairs of 81 are: 1×81; 3×27; 9×9.
Thus, the only two-digit number whose digits have a product of 81 is 99.
The factor pairs of 24 are: 1×24; 2×12; 3×8; 4×6.
Thus, the only two-digit numbers whose digits have a product of 24 are $38, 83, 46, 64$.
The factor pairs of 42 are: 1×42; 3×14; 6×7.
Thus, the only two-digit numbers whose digits have a product of 42 are $67, 76$.
A summary is shown in the table below.

Two-digit number	Preceding number
18	$29, 92, 36, 63$
81	99
24	$38, 83, 46, 64$
42	$67, 76$

Therefore, the only two digit numbers for which the process stops at 8 after 2 steps are $29, 92, 36, 63, 99, 38, 83, 46, 64, 67, 76$.

(c) For the process to stop at 4, the preceding number must have two digits whose product is 4.
The factor pairs of 4 are: 1×4; 2×2.
The only two-digit numbers whose digits have a product of 4 are 14, 41, and 22.
We now find all two-digit numbers whose digits have a product of 14, 41, or 22.

Next, consider the factor pairs of 14, 41 and 22.
The factor pairs of 14 are: 1×14; 2×7.
The only two-digit numbers whose digits have a product of 14 are 27 and 72.
The only factor pair of 41 is 1×41.
Since 1 and 41 cannot be used to form a two-digit number, there are no two-digit numbers whose digits have a product of 41.
The factor pairs of 22 are: 1×22; 2×11.
Thus, there are no two-digit numbers whose digits have a product of 22.
A summary is shown in the table below.

Two-digit number	Preceding number
14	$27, 72$
41	there are none
22	there are none

We now find all two-digit numbers whose digits have a product of 27 or 72.
This is summarized in the table below.

Two-digit number	Preceding number
27	$39, 93$
72	$89, 98$

Since there are no two-digit numbers whose digits have a product of $39, 93, 89$ or 98, we have completed our list.
The only two digit numbers for which the process stops at 4
are $14, 41, 22, 27, 72, 39, 93, 89, 98$.

(d) Although systematic trial and error will find us *the* two-digit number for which the process stops after 4 steps (there is only one), some work done earlier in this question may save us some time.
Notice in part (b) that there are 11 different numbers for which the process stops at 8 after 2 steps.
Of these 11 numbers only 3 can be continued on backwards to previous steps.
These are 36, 63 and 64.
Since there is only one step that precedes 63, that is 79 or 97, we have not found a 4-step number.
Since there is only one step that precedes 64, that is 88, we have not found a 4-step number.
However, 49 precedes 36 and 77 precedes 49.
That is, beginning with 77 the process stops after 4 steps as shown below.

Two-digit number	Step 1	Step 2	Step 3	Step 4
77	$7 \times 7 = 49$	$4 \times 9 = 36$	$3 \times 6 = 18$	$1 \times 8 = 8$

Therefore, 77 is a two-digit number for which the process stops after 4 steps.
Can you show that 77 is the only two-digit number for which the process stops after 4 steps?
Can you show that there is no two-digit number for which the process stops after more than 4 steps?

4. (a) Initially, Ian has 365 two-dollar coins.
That is, Ian begins with $365 \times \$2$ or \$730.
After 365 days, he has spent $365 \times \$1.72$ or \$627.80.
Therefore, Ian will have $\$730 - \$627.80$ or \$102.20 remaining in the coin jar after 365 days.

(b) Since Ian starts with 365 \$2.00 coins and his tea costs \$1.72 each day for 365 days, then he will use at most one of the \$2.00 coins each day. This means that he always has \$2.00 coins left.
Since Ian pays with the least amount possible each day, then he never pays more than \$2.00; he will pay with less than \$2.00 if he has accumulated at least \$1.72 in "loose change" (that is, in coins valued less than \$2.00).
Since Ian pays each time with \$2.00 or less, then he receives at most $\$2.00 - \$1.72 = \$0.28$ in change each time.
Since Ian receives at most \$0.28 in change each time, then he receives at most one 25¢ coin each time.
We will show that the maximum number of 25¢ coins that he can have is 7.
Suppose Ian had 8 or more 25¢ coins in his jar at one time.
In this case, he must have had 7 or more 25¢ coins in his jar on the previous day since he never gets more than one 25¢ coin back in change.
But if Ian had 7 or more 25¢ coins in his jar, then he has at least \$1.75 in loose change and so would give the cashier at most \$1.75 and so would get at most \$0.03 in change.
In this case, he would not get a 25¢ coin as change and so would never get to 8 25¢ coins.
Therefore, Ian can never have more than 7 25¢ coins in his jar.
Lastly, we show that the maximum number of 25¢ coins is indeed 7 by showing that this actually happens.
Note that when Ian gives the cashier a \$2.00 coin and receives \$0.28 in change, the change will come as 1 25¢ coin and 3 1¢ coins.
Starting on the first day, Ian gives the cashier a \$2.00 coin and receives 28¢ in loose change until he has accumulated more than \$1.72 in loose change.
Note that after 6 days, Ian will have received $6 \times \$0.28 = \$1.68$ in loose change (6 25¢ coins and $3 \times 6 = 18$ 1¢ coins).
Since Ian still has less than \$1.72, he must again use a \$2.00 coin on day 7, and so he again receives 28¢ in loose change on day 7, giving him 7 25¢ coins and 21 1¢ coins in total.
Therefore, there is a point at which Ian has 7 25¢ coins and Ian can never have 8 25¢ coins.
Thus, the maximum number of 25¢ coins that he can have is 7.

(c) Part (b) describes Ian giving the cashier a \$2.00 coin and receiving \$0.28 in change, the change coming as 1 25¢ coin and 3 1¢ coins.
This situation happens frequently, so we define it to be a *typical day*.
Typical days occur whenever Ian has less than \$1.72 in loose change and he must offer the cashier a \$2.00 coin as payment for the tea.
For each day listed in the table below, the number of each type of coin in the jar at the end of the day is given.
To condense the table, some typical days have been omitted.
However, on each of these typical days we know that Ian paid with a \$2.00 coin and received 1 25¢ coin and 3 1¢ coins as change.

Coin Type	Day 7	Day 8	Day 14	Day 15	Day 21	Day 22	Day 28	Day 29	Day 35	Day 36	Day 42	Day 43	Day 49	Day 50
\$2.00	358	358	352	352	346	346	340	340	334	334	328	328	322	322
25¢	7	0	6	0	6	0	6	0	6	0	6	0	6	0
1¢	21	24	42	20	38	16	34	12	30	8	26	4	22	0

To clarify the table somewhat, the omitted days 9 to 13 (for example) are typical days.
That is, Ian will offer a \$2.00 coin as payment and the number of \$2.00 coins in the jar will decrease by 1 on each of these days.
Ian will receive exactly 1 25¢ coin and 3 1¢ coins as change on each of these days.
The table shows that beginning at day 8, every 7^{th} day is not a typical day.
That is, on days $8, 15, 22, 29, 36, 43$ and 49, Ian has at least \$1.72 in loose change that he offers the cashier for payment instead of using a \$2.00 coin.
All other days listed in the table and omitted from the table are typical days.
After 50 days, Ian has used exactly $365 - 322 = 43$ \$2.00 coins and has only \$2.00 coins remaining in the coin jar.
This is the first day that the coin jar returns to its initial state of containing only \$2.00 coins.
Since we are starting again on day 51 with only \$2.00 coins, the cycle of coins offered by Ian and received from the cashier will repeat every 50 days.
That is, after 250 days Ian will have used exactly $5 \times 43 = 215$ \$2.00 coins and the coin jar will contain $365 - 215 = 150$ \$2.00 coins only (there will be no 25¢ or 1¢ coins).
Thus, the number of 25¢ and 1¢ coins contained in the jar on the 277^{th} day will be the same as those in the jar on the 27^{th} day.
Using the table above, the number of 25¢ and 1¢ coins on the 28^{th} day is 6 and 34, respectively.
Therefore, the number of 25¢ and 1¢ coins on the 27^{th} day is 5 and 31, respectively, since the 28^{th} day is a typical day.
We also note that there are 341 \$2.00 coins in the jar on the 27^{th} day.
Thus, Ian used $365 - 341 = 24$ \$2.00 coins in the first 27 days.
As a result of the repeating pattern of coins in the jar, 24 \$2.00 coins will be used from the 251^{st} day to the end of the 277^{th} day.
Since there are 150 \$2.00 coins in the jar after 250 days, then there are $150 - 24 = 126$ \$2.00 coins in the jar after 277 days.
After 277 days, there will be 126 \$2.00 coins, 5 25¢ coins, and 31 1¢ coins in the jar.

We can check if our final answer is reasonable.
After 277 days, Ian has spent $277 \times \$1.72 = \$476.44$.
Therefore Ian should have $\$730 - \$476.44 = \$253.56$ remaining in the coin jar.
From the answer given, the coin jar contains $126 \times \$2.00 + 5 \times 25¢ + 31¢ = \$253.56$.
We can be reasonably confident that we have answered this question correctly.

It is worth noting that Ian initially has \$2.00 coins or 200¢ coins at his disposal and the price of each tea is 172¢.
The lowest common multiple of 200 and 172 is 8600 (verify this for yourself).
Can you explain how we may have used this lowest common multiple to find the number of days it takes for the jar to return to its initial state of containing only \$2.00 coins?