Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2010 Hypatia Contest

Friday, April 9, 2010

Solutions

1. (a) The cost to fly is \$0.10 per kilometre plus a \$100 booking fee.
To fly 3250 km from A to B, the cost is $3250 \times 0.10 + 100 = 325 + 100 = \$425$.

(b) Since $\triangle ABC$ is a right-angled triangle, then we may use the Pythagorean Theorem.
Thus, $AB^2 = BC^2 + CA^2$, and so $BC^2 = AB^2 - CA^2 = 3250^2 - 3000^2 = 1\,562\,500$, and $BC = 1250$ km (since $BC > 0$).
Piravena travels a distance of $3250 + 1250 + 3000 = 7500$ km for her complete trip.

(c) To fly from B to C, the cost is $1250 \times 0.10 + 100 = \$225$.
To bus from B to C, the cost is $1250 \times 0.15 = \$187.50$.
Since Piravena chooses the least expensive way to travel, she will bus from B to C.
To fly from C to A, the cost is $3000 \times 0.10 + 100 = \$400$.
To bus from C to A, the cost is $3000 \times 0.15 = \$450$.
Since Piravena chooses the least expensive way to travel, she will fly from C to A.
To check, the total cost of the trip would be $\$425 + \$187.50 + \$400 = \$1012.50$ as required.

2. (a) Substituting $x = 6$, then $f(x) - f(x-1) = 4x - 9$ becomes $f(6) - f(5) = 4 \times 6 - 9$.
Since $f(5) = 18$, then $f(6) - 18 = 24 - 9$ or $f(6) - 18 = 15$ and $f(6) = 33$.

(b) Substituting $x = 5$, then $f(x) - f(x-1) = 4x - 9$ becomes $f(5) - f(4) = 4 \times 5 - 9$.
Since $f(5) = 18$, then $18 - f(4) = 20 - 9$ or $18 - f(4) = 11$ and $f(4) = 7$.
Substituting $x = 4$, then $f(x) - f(x-1) = 4x - 9$ becomes $f(4) - f(3) = 4 \times 4 - 9$.
Since $f(4) = 7$, then $7 - f(3) = 16 - 9$ or $7 - f(3) = 7$ and $f(3) = 0$.

(c) Since $f(5) = 18$, then $2(5^2) + 5p + q = 18$, or $50 + 5p + q = 18$ and so $5p + q = -32$.
Since $f(3) = 0$, then $2(3^2) + 3p + q = 0$, or $18 + 3p + q = 0$ and so $3p + q = -18$.
We solve the system of equations:

$$\begin{aligned} 5p + q &= -32 \\ 3p + q &= -18 \end{aligned}$$

Subtracting the second equation from the first gives $2p = -14$ or $p = -7$.
Substituting $p = -7$ into the first equation gives $5(-7) + q = -32$, or $-35 + q = -32$ and $q = 3$.
Therefore if $f(x) = 2x^2 + px + q$, then $p = -7$ and $q = 3$.

3. (a) Since $\triangle ABE$ is equilateral, then $\angle ABE = 60^\circ$.
Therefore, $\angle PBC = \angle ABC - \angle ABE = 90^\circ - 60^\circ = 30^\circ$.
Since $AB = BC$, then $\triangle ABC$ is a right isosceles triangle and $\angle BAC = \angle BCA = 45^\circ$.
Then, $\angle BCP = \angle BCA = 45^\circ$ and

$$\angle BPC = 180^\circ - \angle PBC - \angle BCP = 180^\circ - 30^\circ - 45^\circ = 105^\circ.$$

(b) *Solution 1*
In $\triangle PBQ$, $\angle PBQ = 30^\circ$ and $\angle BQP = 90^\circ$, thus $\angle BPQ = 60^\circ$.
Therefore, $\triangle PBQ$ is a 30° - 60° - 90° triangle with $PQ : PB : BQ = 1 : 2 : \sqrt{3}$.
Since $\frac{PQ}{BQ} = \frac{1}{\sqrt{3}}$, then $\frac{x}{BQ} = \frac{1}{\sqrt{3}}$ and $BQ = \sqrt{3}x$.

Solution 2
In $\triangle PQC$, $\angle QCP = 45^\circ$ and $\angle PQC = 90^\circ$, thus $\angle CPQ = 45^\circ$.
Therefore, $\triangle PQC$ is isosceles and $QC = PQ = x$.
Since $BC = 4$, then $BQ = BC - QC = 4 - x$.

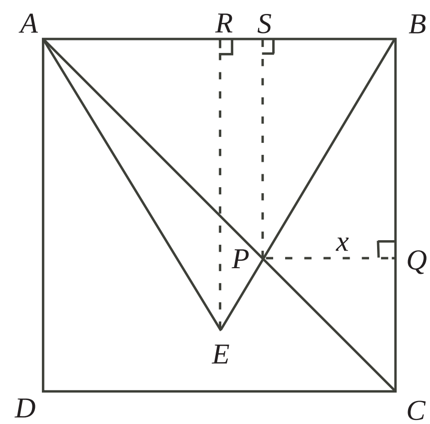
(c) *Solution 1*

In $\triangle PQC$, $\angle QCP = 45^\circ$ and $\angle PQC = 90^\circ$, thus $\angle CPQ = 45^\circ$.
Therefore, $\triangle PQC$ is isosceles and $QC = PQ = x$.
Since $BC = 4$, then $BC = BQ + QC = \sqrt{3}x + x = 4$ or $x(\sqrt{3}+1) = 4$ and $x = \frac{4}{\sqrt{3}+1}$.
Rationalizing the denominator gives $x = \frac{4}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{4(\sqrt{3}-1)}{3-1} = \frac{4(\sqrt{3}-1)}{2} = 2(\sqrt{3}-1)$.

Solution 2

In $\triangle PBQ$, $\angle PBQ = 30^\circ$ and $\angle BQP = 90^\circ$, thus $\angle BPQ = 60^\circ$.
Therefore, $\triangle PBQ$ is a 30°-60°-90° triangle with $PQ : PB : BQ = 1 : 2 : \sqrt{3}$.
Since $\frac{PQ}{BQ} = \frac{1}{\sqrt{3}}$, then $\frac{x}{4-x} = \frac{1}{\sqrt{3}}$ or $\sqrt{3}x = 4 - x$ or $\sqrt{3}x + x = 4$, and $x(\sqrt{3}+1) = 4$ so $x = \frac{4}{\sqrt{3}+1}$.
Rationalizing the denominator gives $x = \frac{4}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{4(\sqrt{3}-1)}{3-1} = \frac{4(\sqrt{3}-1)}{2} = 2(\sqrt{3}-1)$.

(d) *Solution 1*



We adopt the notation $|\triangle XYZ|$ to represent the area of triangle XYZ.
Then, $|\triangle APE| = |\triangle ABE| - |\triangle ABP|$.
Since $\triangle ABE$ is equilateral, $BE = EA = AB = 4$ and the altitude from E to AB bisects side AB at R as shown.
Thus, $AR = RB = 2$ and by the Pythagorean Theorem $ER^2 = BE^2 - RB^2 = 4^2 - 2^2 = 12$ or $ER = \sqrt{12} = 2\sqrt{3}$, since $ER > 0$.
Therefore, the area of $\triangle ABE$ is $\frac{1}{2}(AB)(ER)$, or $\frac{1}{2}(4)(2\sqrt{3}) = 4\sqrt{3}$.
In $\triangle ABP$, construct the altitude from P to S on AB.
Then $PS \perp AB$ and $QB \perp AB$, so $PS \parallel QB$.
Also, $SB \perp QB$ and $PQ \perp QB$, so $SB \parallel PQ$.
Thus, $SBQP$ is a rectangle and $PS = QB$.
From (b) and (c), $QB = 4 - x = 4 - 2(\sqrt{3}-1) = 6 - 2\sqrt{3}$.
Therefore, $|\triangle ABP| = \frac{1}{2}(AB)(PS) = \frac{1}{2}(4)(6-2\sqrt{3}) = 2(6-2\sqrt{3}) = 12 - 4\sqrt{3}$.
Then, $|\triangle APE| = 4\sqrt{3} - (12 - 4\sqrt{3}) = 4\sqrt{3} - 12 + 4\sqrt{3} = 8\sqrt{3} - 12$.

Solution 2

We adopt the notation $|\triangle XYZ|$ to represent the area of triangle XYZ.
Then, $|\triangle APE| = |\triangle ABE| - |\triangle ABP|$.
However, $|\triangle ABP| = |\triangle ABC| - |\triangle BPC|$.
Thus, $|\triangle APE| = |\triangle ABE| - (|\triangle ABC| - |\triangle BPC|) = |\triangle ABE| + |\triangle BPC| - |\triangle ABC|$.
Since $\triangle ABE$ is equilateral, $BE = EA = AB = 4$ and the altitude from E to AB bisects side AB at R as shown.
Thus, $\triangle ERB$ is a 30°-60°-90° triangle with $ER : RB = \sqrt{3} : 1$ or $ER = (RB)\sqrt{3} = 2\sqrt{3}$.
Therefore, $|\triangle ABE| = \frac{1}{2}(AB)(ER) = \frac{1}{2}(4)(2\sqrt{3}) = 4\sqrt{3}$.
Since PQ is an altitude of $\triangle BPC$, $|\triangle BPC| = \frac{1}{2}(BC)(PQ) = \frac{1}{2}(4)(2\sqrt{3}-2) = 4\sqrt{3} - 4$.
In triangle ABC, $\angle ABC = 90^\circ$.
Thus, $|\triangle ABC| = \frac{1}{2}(AB)(BC) = \frac{1}{2}(4)(4) = 8$.
Thus, $|\triangle APE| = |\triangle ABE| + |\triangle BPC| - |\triangle ABC| = (4\sqrt{3}) + (4\sqrt{3} - 4) - 8 = 8\sqrt{3} - 12$.

4. (a) We solve by factoring,

$$\begin{aligned} x^4 - 6x^2 + 8 &= 0 \\ (x^2-4)(x^2-2) &= 0 \end{aligned}$$

Therefore, $x^2 = 4$ or $x^2 = 2$, and so $x = \pm 2$ or $x = \pm\sqrt{2}$.

The real values of x satisfying $x^4 - 6x^2 + 8 = 0$ are $x = -2, 2, -\sqrt{2}$, and $\sqrt{2}$.

(b) We want the smallest positive integer N for which,

$$\begin{aligned} x^4 + 2010x^2 + N &= (x^2+rx+s)(x^2+tx+u) \\ x^4 + 2010x^2 + N &= x^4 + tx^3 + ux^2 + rx^3 + rtx^2 + rux + sx^2 + stx + su \\ x^4 + 2010x^2 + N &= x^4 + tx^3 + rx^3 + ux^2 + rtx^2 + sx^2 + rux + stx + su \\ x^4 + 2010x^2 + N &= x^4 + (t+r)x^3 + (u+rt+s)x^2 + (ru+st)x + su \end{aligned}$$

Equating the coefficients from the left and right sides of this equation we have,
$t + r = 0$, $u + rt + s = 2010$, $ru + st = 0$, and $su = N$.
From the first equation we have $t = -r$.
If we substitute $t = -r$ into the third equation, then $ru - rs = 0$ or $r(u - s) = 0$.
Since $r \neq 0$, then $u - s = 0$ or $u = s$.
Thus, from the fourth equation we have $N = su = u^2$.
That is, to minimize N we need to minimize u^2.
If we substitute $t = -r$ and $s = u$ into the second equation, then $u + rt + s = 2010$ becomes $u + r(-r) + u = 2010$ or $2u - r^2 = 2010$ and so $u = \dfrac{2010 + r^2}{2}$.
Thus, $u > 0$. So to minimize u^2, we minimize u or equivalently, we minimize r.
Since u and r are integers and $r \neq 0$, u is minimized when $r = \pm 2$ (r must be even) or $u = \dfrac{2014}{2} = 1007$.
Therefore, the smallest positive integer N for which $x^4 + 2010x^2 + N$ can be factored as $(x^2 + rx + s)(x^2 + tx + u)$ with r, s, t, u integers and $r \neq 0$ is $N = u^2 = 1007^2 = 1014049$.

(c) Replacing the coefficient 2010 with M in part (b) and again equating coefficients, we have the similar four equations $t + r = 0$, $u + rt + s = M$, $ru + st = 0$, and $su = N$.
Thus we have,

$$\begin{aligned} N - M &= su - (u + rt + s) \\ 37 &= u^2 - (2u - r^2) \\ 37 &= u^2 - 2u + r^2 \\ 37 + 1 &= u^2 - 2u + 1 + r^2 \\ 38 &= (u-1)^2 + r^2 \end{aligned}$$

and so $r = \pm\sqrt{38 - (u-1)^2}$.
In the table below we attempt to find integer solutions for u and r:

u	$(u-1)^2$	r
1	0	$\pm\sqrt{38}$
0 or 2	1	$\pm\sqrt{37}$
-1 or 3	4	$\pm\sqrt{34}$
-2 or 4	9	$\pm\sqrt{29}$
-3 or 5	16	$\pm\sqrt{22}$
-4 or 6	25	$\pm\sqrt{13}$
-5 or 7	36	$\pm\sqrt{2}$

We see that for all choices of u above, r is not an integer.
For any other integer choice of u not listed, $(u-1)^2 > 38$ and then $38-(u-1)^2 < 0$, so there are no real solutions for r.
Thus, when u is an integer, r cannot be, so u and r cannot both be integers. Therefore, x^4+Mx^2+N cannot be factored as in (b) for any integers M and N with $N-M=37$.
Note: Alternatively, we could have stated that $(u-1)^2+r^2$ represents the sum of two perfect squares. Since no pair of perfect squares (from the list $0, 1, 4, 9, 16, 25, 36$) sums to 38, then $(u-1)^2+r^2 \neq 38$ for any integers u and r.