Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2008 Cayley Contest

(Grade 10)

Tuesday, February 19, 2008

Solutions

1. Calculating, $3^2 - 2^2 + 1^2 = 9 - 4 + 1 = 6$.

   Answer: (E)

2. Calculating, $\dfrac{\sqrt{25-16}}{\sqrt{25}-\sqrt{16}} = \dfrac{\sqrt{9}}{5-4} = \dfrac{3}{1} = 3$.

   Answer: (B)

3. In decimal form, the five possible answers are

   $$0.75, 1.2, 0.81, 1.333\ldots, 0.7$$

   The differences between 1 and these possiblities are

   $1-0.75 = 0.25 \qquad 1.2-1 = 0.2 \qquad 1-0.81 = 0.19 \qquad 1.333\ldots-1 = 0.333\ldots \qquad 1-0.7 = 0.3$

   The possibility with the smallest difference with 1 is 0.81, so 0.81 is closest to 1.

   Answer: (C)

4. In total, there are $5 + 6 + 7 + 8 = 26$ jelly beans in the bag.
   Since there are 8 blue jelly beans, the probability of selecting a blue jelly bean is $\frac{8}{26} = \frac{4}{13}$.

   Answer: (D)

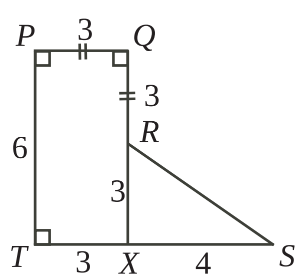
5. Since $52\,28\square$ is a multiple of 6, then it must be a multiple of 2 and a multiple of 3.
   Since it is a multiple of 2, the digit represented by $\square$ must be even.
   Since it is a multiple of 3, the sum of its digits is divisible by 3.
   The sum of its digits is $5 + 2 + 2 + 8 + \square = 17 + \square$.
   Since $\square$ is even, the possible sums of digits are 17, 19, 21, 23, 25 (for the possible values 0, 2, 4, 6, 8 for $\square$).
   Of these possibilities, only 21 is divisible by 3, so $\square$ must equal 4.
   We can check that 52 284 is divisible by 6.
   (An alternate approach would have been to use a calculator and test each of the five possible values for $\square$ by dividing the resulting values of $52\,28\square$ by 6.)

   Answer: (C)

6. Since $\dfrac{40}{x} - 1 = 19$, then $\dfrac{40}{x} = 20$.
   Thus, $x = 2$, since the number that we must divide 40 by to get 20 is 2.

   Answer: (D)

7. We extend QR to meet TS at X.
   Since $PQ = QR$, then $QR = 3$.
   Since $PQXT$ has three right angles, it must be a rectangle, so $TX = PQ = 3$.
   Also, $QX = PT = 6$.
   Since $TS = 7$ and $TX = 3$, then $XS = TS - TX = 7 - 3 = 4$.
   Since $QX = 6$ and $QR = 3$, then $RX = QX - QR = 6 - 3 = 3$.



Since $PQXT$ is a rectangle, then $\angle RXS = 90^\circ$.
By the Pythagorean Theorem in $\triangle RXS$,

$$RS^2 = RX^2 + XS^2 = 3^2 + 4^2 = 9 + 16 = 25$$

so $RS = 5$, since $RS > 0$.
Therefore, the perimeter is $PQ + QR + RS + ST + TP = 3 + 3 + 5 + 7 + 6 = 24$.

Answer: (A)

8. Since $PQ = QR$, then $\angle QPR = \angle QRP$.
Since $\angle PQR + \angle QPR + \angle QRP = 180^\circ$, then $40^\circ + 2(\angle QRP) = 180^\circ$, so $2(\angle QRP) = 140^\circ$ or $\angle QRP = 70^\circ$.
Since $\angle PRQ$ and $\angle SRT$ are opposite angles, then $\angle SRT = \angle PRQ = 70^\circ$.
Since $RS = RT$, then $\angle RST = \angle RTS = x^\circ$.
Since $\angle SRT + \angle RST + \angle RTS = 180^\circ$, then $70^\circ + 2x^\circ = 180^\circ$ or $2x = 110$ or $x = 55$.

Answer: (C)

9. *Solution 1*
Since a and b are both odd, then ab is odd.
Therefore, the largest even integer less than ab is $ab - 1$.
Since every other positive integer less than or equal to $ab - 1$ is even, then the number of even positive integers less than or equal to $ab - 1$ (thus, less than ab) is $\dfrac{ab-1}{2}$.

Solution 2
Since $a = 7$ and $b = 13$, then $ab = 91$.
The even positive integers less than $ab = 91$ are $2, 4, 6, \ldots, 90$.
There are $90 \div 2 = 45$ such integers.
Using $a = 7$ and $b = 13$, the five possible answers are

$$\frac{ab-1}{2} = 45 \qquad \frac{ab}{2} = \frac{91}{2} \qquad ab - 1 = 90 \qquad \frac{a+b}{4} = 5 \qquad (a-1)(b-1) = 72$$

Therefore, the answer must be $\dfrac{ab-1}{2}$.

Answer: (A)

10. For her 200 daytime minutes, Vivian is charged $200 \times \$0.10 = \$20$.
Since Vivian used 300 evening minutes and has 200 free evening minutes, then she is charged for $300 - 200 = 100$ of these minutes, and so is charged $100 \times \$0.05 = \$5$.
Her total bill is thus $\$20 + \$20 + \$5 = \$45$.

Answer: (C)

11. Lex has 265 cents in total.
Since a quarter is worth 25 cents, the total value in cents of the quarters that Lex has is a multiple of 25, and so must end in 00, 25, 50 or 75.
Since the remaining part of the 265 cents is made up of dimes only, the remaining part is a multiple of 10, so ends in a 0.
Thus, the value of the quarters must end with a 5, so ends with 25 or 75.
Since Lex has more quarters than dimes, we start by trying to determine the largest possible number of quarters that he could have.

The largest possible value of his quarters is thus 225 cents, which would be $225 \div 25 = 9$ quarters, leaving $265 - 225 = 40$ cents in dimes, or 4 dimes.
Thus, Lex has $9 + 4 = 13$ coins in total.
(The next possible largest value of his quarters is 175 cents, which would come from 7 quarters and so 90 cents in dimes or 9 dimes. This does not satisfy the condition that the number of quarters is larger than the number of dimes.)

ANSWER: (B)

12. *Solution 1*
Since $\angle OMP$ and $\angle GMH$ are opposite angles, then $\angle OMP = \angle GMH$.
The x-axis and the y-axis are perpendicular, so $\angle POM = 90^\circ$, so $\angle POM = \angle GHM$.
Since M is the midpoint of OH, then $OM = HM$.
Therefore, $\triangle POM$ and $\triangle GHM$ are congruent by Angle-Side-Angle.
Thus, $GH = OP = 4$.
Since GH is perpendicular to the x-axis, then the x-coordinate of G is 12, so G has coordinates $(12, 4)$.

Solution 2
Since GH is perpendicular to the x-axis, then the x-coordinate of G is 12, so G has coordinates $(12, g)$, for some value of g.
Since $OH = 12$ and M is the midpoint of OH, then $OM = \frac{1}{2}(12) = 6$, so M has coordinates $(6, 0)$.
To get from P to M, we move 6 units right and 4 units up.
To get from M to G, we also move 6 units right. Since P, M and G lie on a line, then we must also move 4 units up to get from M to G.
Therefore, the coordinates of G are $(12, 4)$.

ANSWER: (E)

13. In the given layout, the white face and the face containing the "U" are joined so that the "U" opens towards the edge joining these faces. Therefore, (A) cannot be correct as the "U" does not open towards the white face across the common edge.
The given layout shows that the white face and the grey face cannot be joined along along an edge when folded. Therefore, (C) cannot be correct.
The given layout also shows that the face containing the "U" and the face containing the "V" cannot be joined along along an edge when folded. Therefore, (D) cannot be correct.
In the given layout, the grey face and the face containing the "U" are joined so that the "U" opens away from the edge joining these faces. Therefore, (E) cannot be correct as the "U" does not open away from the grey face across the common edge.
Having eliminated the other possibilities, the correct answer must be (B). (We can check by visualizing the folding process that this cube can indeed be made.)

ANSWER: (B)

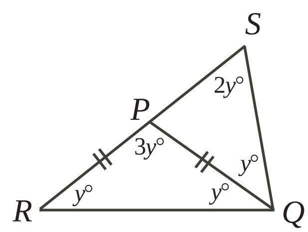
14. The third term is odd ($t = 5$), so the fourth term is $3(5) + 1 = 16$, which is even.
Thus, the fifth term is $\frac{1}{2}(16) = 8$, which is even.
Thus, the sixth term is $\frac{1}{2}(8) = 4$, which is even.
Thus, the seventh term is $\frac{1}{2}(4) = 2$, which is even.
Thus, the eighth term is $\frac{1}{2}(2) = 1$, which is odd.
Thus, the ninth term is $3(1) + 1 = 4$, which is even.
Thus, the tenth term is $\frac{1}{2}(4) = 2$.

ANSWER: (A)

15. First, we find the prime factors of 555.
Since 555 ends with a 5, it is divisible by 5, with $555 = 5 \times 111$.
Since the sum of the digits of 111 is 3, then 111 is divisible by 3, with $111 = 3 \times 37$.
Therefore, $555 = 3 \times 5 \times 37$, and each of 3, 5 and 37 is a prime number.
The possible ways to write 555 as the product of two integers are 1×555, 3×185, 5×111, and 15×37. (In each of these products, two or more of the prime factors have been combined to give a composite divisor.)
The only pair where both members are two-digit positive integers is 37 and 15, so $x + y$ is $37 + 15 = 52$.

Answer: (A)

16. *Solution 1*
Since RPS is a straight line, then $\angle SPQ = 180^\circ - \angle RPQ = 180^\circ - 3y^\circ$.
Using the angles in $\triangle PQS$, we have $\angle PQS + \angle QSP + \angle SPQ = 180^\circ$.
Thus, $x^\circ + 2y^\circ + (180^\circ - 3y^\circ) = 180^\circ$ or $x - y + 180 = 180$ or $x = y$.
(We could have instead looked at $\angle RPQ$ as being an external angle to $\triangle SPQ$.)
Since $x = y$, then $\angle RQS = 2y^\circ$.
Since $RP = PQ$, then $\angle PRQ = \angle PQR = x^\circ = y^\circ$.



Therefore, the angles of $\triangle RQS$ are y°, $2y^\circ$ and $2y^\circ$.
Thus, $y^\circ + 2y^\circ + 2y^\circ = 180^\circ$ or $5y = 180$ or $y = 36$.
Therefore, $\angle RPQ = 3y^\circ = 3(36)^\circ = 108^\circ$.

Solution 2
Since $RP = PQ$, then $\angle PRQ = \angle PQR = x^\circ$.
Looking at the sum of the angles in $\triangle RPQ$ and $\triangle RSQ$, we have $x^\circ + 3y^\circ + x^\circ = 180^\circ$ (or $2x + 3y = 180$) and $x^\circ + 2y^\circ + 2x^\circ = 180$ (or $3x + 2y = 180$).
Adding these two equations gives $5x + 5y = 360$ or $x + y = \frac{1}{5}(360) = 72$.
Thus, $2x + 2y = 2(72) = 144$, so $2x + 3y = 180$, gives $y = 180 - (2x + 2y) = 180 - 144 = 36$.
Therfore, $\angle RPQ = 3y^\circ = 3(36)^\circ = 108^\circ$.

Answer: (B)

17. The maximum possible value of $\dfrac{p}{q}$ is when p is as large as possible (that is, 10) and q is as small as possible (that is, 12). Thus, the maximum possible value of $\dfrac{p}{q}$ is $\dfrac{10}{12} = \dfrac{5}{6}$.
The minimum possible value of $\dfrac{p}{q}$ is when p is as small as possible (that is, 3) and q is as large as possible (that is, 21). Thus, the maximum possible value of $\dfrac{p}{q}$ is $\dfrac{3}{21} = \dfrac{1}{7}$.
The difference between these two values is $\dfrac{5}{6} - \dfrac{1}{7} = \dfrac{35}{42} - \dfrac{6}{42} = \dfrac{29}{42}$.

Answer: (A)

18. Suppose that there are x \$1 bills.
Thus, there are $(x + 11)$ \$2 bills and $(x - 18)$ \$3 bills.
Since the total value of the money is \$100, then

$$\begin{aligned} 1(x) + 2(x + 11) + 3(x - 18) &= 100 \\ x + 2x + 22 + 3x - 54 &= 100 \\ 6x - 32 &= 100 \\ 6x &= 132 \\ x &= 22 \end{aligned}$$

Therefore, there are 22 \$1 bills.

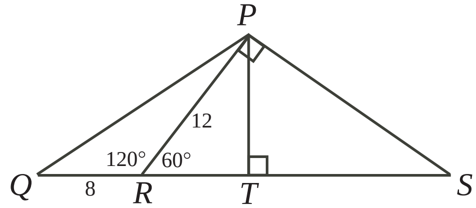
Answer: (C)

19. *Solution 1*
Since $\frac{2}{3}$ of the apples are rotten, $\frac{3}{4}$ of the pears are rotten, and the number of rotten apples and pears are equal, then we could try 6 rotten pieces of each type of fruit. (We choose 6 as it is a multiple of the numerator of each fraction.)
If there are 6 rotten apples, then the total number of apples is $\frac{3}{2}(6) = 9$.
If there are 6 rotten pears, then the total number of pears is $\frac{4}{3}(6) = 8$.
Therefore, there are $9 + 8 = 17$ pieces of fruit in total, of which $6 + 6 = 12$ are rotten.
Thus, $\frac{12}{17}$ of the fruit are rotten.

Solution 2
Suppose there are a apples and p pears in total.
Since the number of rotten apples and rotten pears are equal, then $\frac{2}{3}a = \frac{3}{4}p$, so $p = \frac{4}{3}(\frac{2}{3}a) = \frac{8}{9}a$.
Therefore, the total number of pieces of fruit is $a + p = a + \frac{8}{9}a = \frac{17}{9}a$.
Also, the total number of rotten fruit is $2(\frac{2}{3}a) = \frac{4}{3}a$, so the fraction of the total amount of fruit that is rotten is $\frac{\frac{4}{3}a}{\frac{17}{9}a} = \frac{4}{3} \cdot \frac{9}{17} = \frac{12}{17}$.

Answer: (D)

20. Since $\angle QRP = 120^\circ$ and QRS is a straight line, then $\angle PRS = 180^\circ - 120^\circ = 60^\circ$.
Since $\angle RPS = 90^\circ$, then $\triangle SRP$ is a 30°-60°-90° triangle.
Therefore, $RS = 2PR = 2(12) = 24$.
Drop a perpendicular from P to T on RS.



Since $\angle PRT = 60^\circ$ and $\angle PTR = 90^\circ$, then $\triangle PRT$ is also a 30°-60°-90° triangle.
Therefore, $PT = \frac{\sqrt{3}}{2}PR = 6\sqrt{3}$.
Consider $\triangle QPS$. We may consider QS as its base with height PT.
Thus, its area is $\frac{1}{2}(6\sqrt{3})(8 + 24) = 96\sqrt{3}$.

Answer: (E)

21. Since the radius of the inner circular pane is 20 cm, then its area is $\pi 20^2 = 400\pi$ cm^2.
Therefore, the area of each of the outer panes is also 400π cm^2, so the total area of the circular window is $9(400) = 3600\pi$ cm^2.
If the radius of the larger circle is R, then $\pi R^2 = 3600\pi$, so $R^2 = 3600$ or $R = 60$, since $R > 0$.
Since each of the outer lines can be extended to form a radius by joining its inner end to the centre using a radius of the inner circle, then the radius of the larger circle is $x + 20 = 60$, so $x = 40 = 40.0$, to the nearest tenth.
(It was not actually necessary to calculate the area of the circles. Since the large circle is formed from 9 pieces of equal area, its area is 9 times that of the inner circle. Thus, its radius is $\sqrt{9} = 3$ times that of the inner circle.)

Answer: (A)

22. There are 52 terms in the sum: the number 1, the number 11, and the 50 numbers starting with a 1, ending with a 1 and with 1 to 50 zeroes in between. The longest of these terms thus has 52 digits (50 zeroes and 2 ones).
When the units digits of all 52 terms are added up, their sum is 52, so the units digit of N is 2, and a 5 carried to the tens digit.
In the tens digit, there is only 1 non-zero digit: the 1 in the number 11. Therefore, using the carry, the tens digit of N is $1 + 5 = 6$.
In each of positions 3 to 52 from the right-hand end, there is only one non-zero digit, which is a 1.
Therefore, the digit in each of these positions in N is also a 1. (There is no carrying to worry about.)
Therefore, $N = 11\cdots1162$, where N has $52 - 2 = 50$ digits equal to 1.
This tells us that the sum of the digits of N is $50(1) + 6 + 2 = 58$.

Answer: (D)

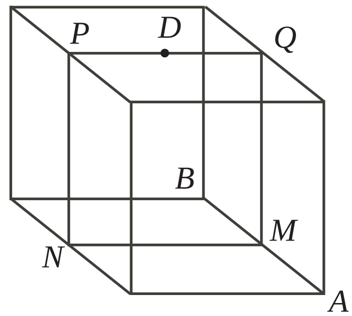
23. The number 4^3 equals 64.
To express 64 as a^b where a and b are integers, we can use 64^1, 8^2, 4^3, 2^6, $(-2)^6$, and $(-8)^2$.
We make a table to evaluate x and y:

$y-1$	$x+y$	y	x
64	1	65	-64
8	2	9	-7
4	3	5	-2
2	6	3	3
-2	6	-1	7
-8	2	-7	9

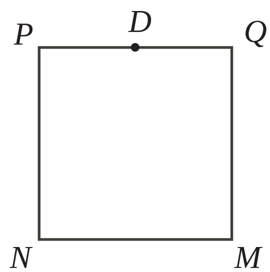
Therefore, there are 6 possible values for x.

Answer: (E)

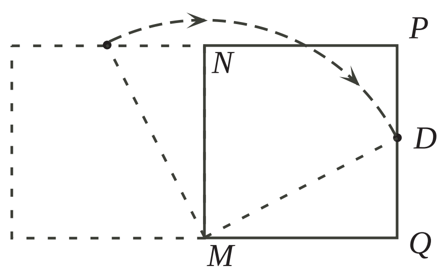
24. Suppose the cube rolls first over edge AB.
Consider the cube as being made up of two half-cubes (each of dimensions $1 \times 1 \times \frac{1}{2}$) glued together at square $PQMN$. (Note that $PQMN$ lies on a vertical plane.)
Since dot D is in the centre of the top face, then D lies on square $PQMN$.



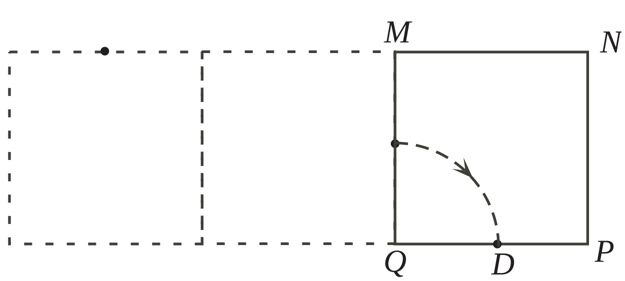
Since the cube always rolls in a direction perpendicular to AB, then the dot will always roll in the plane of square $PQMN$.



So we can convert the original three-dimensional problem to a two-dimensional problem of this square slice rolling.
Square $MNPQ$ has side length 1 and $DQ = \frac{1}{2}$, since D was in the centre of the top face.
By the Pythagorean Theorem, $MD^2 = DQ^2 + QM^2 = \frac{1}{4} + 1 = \frac{5}{4}$, so $MD = \frac{\sqrt{5}}{2}$ since $MD > 0$.
In the first segment of the roll, we start with NM on the table and roll, keeping M stationary, until Q lands on the table.



This is a rotation of 90° around M. Since D is at a constant distance of $\frac{\sqrt{5}}{2}$ from M, then D rotates along one-quarter (since 90° is $\frac{1}{4}$ of 360°) of a circle of radius $\frac{\sqrt{5}}{2}$, for a distance of $\frac{1}{4}(2\pi\frac{\sqrt{5}}{2}) = \frac{\sqrt{5}}{4}\pi$.
In the next segment of the roll, Q stays stationary and the square rolls until P touches the table.



Again, the roll is one of 90°. Note that $QD = \frac{1}{2}$. Thus, again D moves through one-quarter of a circle this time of radius $\frac{1}{2}$, for a distance of $\frac{1}{4}(2\pi\frac{1}{2}) = \frac{1}{4}\pi$.
Through the next segment of the roll, P stays stationary and the square rolls until N touches the table. This is similar to the second segment, so D rolls through a distance of $\frac{1}{4}\pi$.
Through the next segment of the roll, N stays stationary and the square rolls until M touches the table. This will be the end of the process as the square will end up in its initial position. This segment is similar to the first segment so D rolls through a distance of $\frac{\sqrt{5}}{4}\pi$.
Therefore, the total distance through which the dot travels is $\frac{\sqrt{5}}{4}\pi + \frac{1}{4}\pi + \frac{1}{4}\pi + \frac{\sqrt{5}}{4}\pi$ or $\left(\dfrac{1+\sqrt{5}}{2}\right)\pi$.

Answer: (E)

25. There are $7! = 7(6)(5)(4)(3)(2)(1)$ possible arrangements of the 7 numbers $\{1, 2, 3, 11, 12, 13, 14\}$.
To determine the average value of

$$(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-e)^2 + (e-f)^2 + (f-g)^2 \qquad (*)$$

we determine the sum of the values of this expression over all possible arrangements, and then divide by the number of arrangements.
Let x and y be two of these seven numbers.
In how many of these arrangements are x and y adjacent?
Treat x and y as a single unit $(x\ y)$ with 5 other numbers to be placed on either side of, but not between, $x\ y$.
This gives 6 "numbers" ($x\ y$ and 5 others) to arrange, which can be done in $6(5)(4)(3)(2)(1)$ or $6!$ ways.
But y could be followed by x, so there are $2(6!)$ arrangements with x and y adjacent, since there are the same number of arrangements with x followed by y as there are with y followed by x.
Therefore, when we add up the values of (*) over all possible arrangements, the term $(x-y)^2$ (which is equal to $(y-x)^2$) will occur $2(6!)$ times.
This is true for any pair x and y.
Therefore, the average value is $2(6!)$ times the sum of $(x-y)^2$ over all choices of x and y with $x < y$, divided by $7!$.
The sum of all possible values of $(x-y)^2$ is

$$\begin{array}{ccccccccccccccc}
1^2 & + & 2^2 & + & 10^2 & + & 11^2 & + & 12^2 & + & 13^2 & & \\
 & + & 1^2 & + & 9^2 & + & 10^2 & + & 11^2 & + & 12^2 & & \\
 & & & + & 8^2 & + & 9^2 & + & 10^2 & + & 11^2 & & \\
 & & & & & + & 1^2 & + & 2^2 & + & 3^2 & & \\
 & & & & & & & + & 1^2 & + & 2^2 & & \\
 & & & & & & & & & + & 1^2 & = & 1372
\end{array}$$

(Here, we have paired 1 with each of the 6 larger numbers, then 2 with each of the 5 larger numbers, and so on. We only need to pair each number with all of the larger numbers because we have accounted for the reversed pairs in our method above.)
Therefore, the average value is $\dfrac{2(6!)(1372)}{7!} = \dfrac{2(1372)}{7} = 392$.

Answer: (D)