Canadian
Mathematics
Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2002 Solutions

Cayley Contest (Grade 10)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

1. Expanding and simplifying,

$$5x+2(4+x)=5x+(8+2x)$$
$$=7x+8$$

ANSWER: (C)

2. Evaluating,

$$(2+3)^2-\left(2^2+3^2\right)=5^2-(4+9)$$
$$=25-13$$
$$=12$$

ANSWER: (A)

3. If $x=-3$,

$$x^2-4(x-5)=(-3)^2-4(-3-5)$$
$$=9-4(-8)$$
$$=41$$

ANSWER: (D)

4. Since $n=\frac{5}{6}(240)$, then $\frac{2}{5}n=\frac{2}{5}\left(\frac{5}{6}\right)(240)=\frac{1}{3}(240)=80$. ANSWER: (B)

5. Using exponent laws,

$$2^{-2}\times2^{-1}\times2^{0}\times2^{1}\times2^{2}=2^{-2-1+0+1+2}=2^0=1.$$

Alternatively,

$$2^{-2}\times2^{-1}\times2^{0}\times2^{1}\times2^{2}=\tfrac{1}{4}\times\tfrac{1}{2}\times1\times2\times4=1.$$

ANSWER: (B)

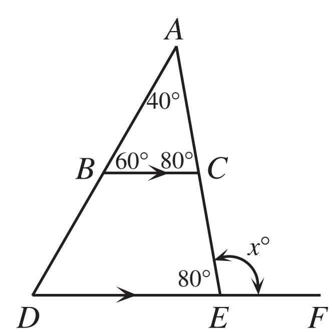
6. In ΔABC,

$$\angle ACB+40^\circ+60^\circ=180^\circ$$
$$\angle ACB=80^\circ$$

Since BC is parallel to DE, $\angle AED=\angle ACB=80^\circ$.

So

$$x^\circ=180^\circ-\angle AED=100^\circ$$
$$x=100$$



ANSWER: (D)

7. Since the line has slope $\frac{1}{2}$, then for every 2 units we move to the right, the line rises by 1 unit. Therefore, $(-2+2,4+1)=(0,5)$ lies on the line. Thus the y-intercept is 5.

ANSWER: (A)

8. Since Megan's scoring average was 18 points per game after 3 games, she must have scored $3 \times 18 = 54$ points over her first three games. Similarly, she must have scored $4 \times 17 = 68$ points over her first 4 games. So in the fourth game, she scored $68 - 54 = 14$ points.

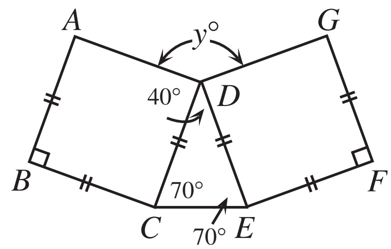
ANSWER: (E)

9. Since squares $ABCD$ and $DEFG$ have equal side lengths, then $DC = DE$, ie. ΔCDE is isosceles. Therefore, $\angle DEC = \angle DCE = 70^\circ$ and so

$$\angle CDE = 180^\circ - 70^\circ - 70^\circ = 40^\circ$$

and

$$y^\circ = 360^\circ - \angle ADC - \angle CDE - \angle GDE$$
$$y^\circ = 360^\circ - 90^\circ - 40^\circ - 90^\circ$$
$$y^\circ = 140^\circ$$
$$y = 140$$



ANSWER: (E)

10. Let the original number be x. Then

$$\frac{x-5}{4} = \frac{x-4}{5}$$
$$5(x-5) = 4(x-4)$$
$$5x - 25 = 4x - 16$$
$$x = 9$$

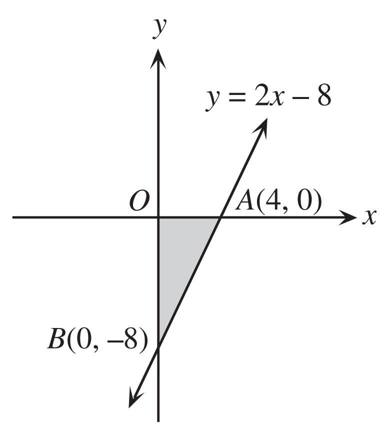
ANSWER: (C)

11. Point B is the y-intercept of $y = 2x - 8$, and so has coordinates $(0,-8)$ from the form of the line. Point A is the x-intercept of $y = 2x - 8$, so we set $y = 0$ and obtain

$$0 = 2x - 8$$
$$2x = 8$$
$$x = 4$$

and so A has coordinates $(4,0)$. This tells us that

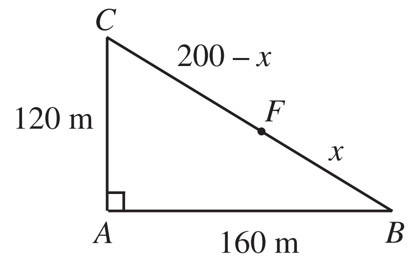
$$\text{Area of } \Delta AOB = \tfrac{1}{2}(OA)(OB) = \tfrac{1}{2}(4)(8) = 16.$$



ANSWER: (B)

12. After the price is increased by 40%, the new price is 140% of \$10.00, or \$14.00. Then after this price is reduced by 30%, the final price is 70% of \$14.00, which is $0.7(\$14.00) = \$9.80$.

ANSWER: (A)



13. By Pythagoras,

$$BC^2 = 120^2 + 160^2 = 40\,000$$

so $BC = 200$ m.

Let $FB = x$. Then $CF = 200 - x$. From the given information,

$$AC + CF = AB + BF$$
$$120 + 200 - x = 160 + x$$
$$160 = 2x$$
$$x = 80$$

Thus, the distance from F to B is 80 m.

ANSWER: (D)

14. *Solution 1*

We factor out a common factor of a from the first two terms and a common factor of b from the last two terms, and so

$$ac + ad + bc + bd = 68$$
$$a(c+d) + b(c+d) = 68$$
$$a(4) + b(4) = 68$$
$$4(a+b) = 68$$
$$a + b = 17$$

using the fact that $c + d = 4$.

Then $a + b + c + d = (a+b) + (c+d) = 17 + 4 = 21$.

Solution 2

We let $c = d = 2$, since $c + d = 4$, and then substitute these values to find that

$$ac + ad + bc + bd = 68$$
$$2a + 2a + 2b + 2b = 68$$
$$a + b = 17$$

Thus $(a+b) + (c+d) = 17 + 4 = 21$.

ANSWER: (D)

15. From A, we can travel to D, E or B.

If we travel $A \to D$, we must then go $D \to E \to F$, following the arrows.

If we travel $A \to E$, we must then go $E \to F$.

If we travel $A \to B$, we can then travel from B to E, C or F.

From E or C, we must travel directly to F.

Thus, there are 5 different paths from A to F.

ANSWER: (B)

16. Let the four consecutive positive integers be n, $n+1$, $n+2$, $n+3$.

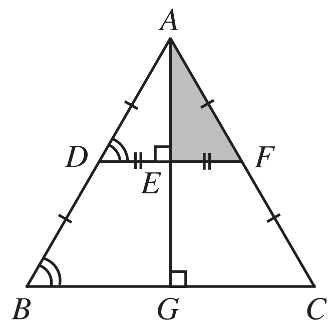
So we need to solve the equation $n(n+1)(n+2)(n+3) = 358\ 800$.
Now $n(n+1)(n+2)(n+3)$ is approximately equal to n^4, so n^4 is close to $358\ 800$, and so n is close to $\sqrt[4]{358\ 800} \approx 24.5$.
So we try $24(25)(26)(27) = 421\ 200$ which is too big, and so we try $23(24)(25)(26) = 358\ 800$, which is the value we want.
So the sum is $23 + 24 + 25 + 26 = 98$.

ANSWER: (B)

17. A "double-single" number is of the form *aab*, where *a* and *b* are different digits. There are 9 possibilities for *a* (since *a* cannot be 0). For each of these possibilities, there are 9 possibilities for *b* (since *b* can be any digit from 0 to 9, provided that it doesn't equal *a*). So there are $9 \times 9 = 81$ possibilities in total.

ANSWER: (A)

18. *Solution 1*



Since ΔADF and ΔABC share $\angle DAF$ and

$$\frac{AD}{AB} = \frac{AF}{AC} = \frac{1}{2}$$

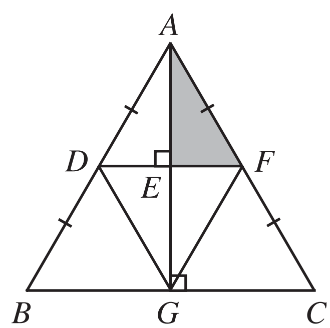
then ΔADF is similar to ΔABC. So the ratio of their areas is the square of the ratio of their side lengths, that is

Area of $\Delta ADF = \left(\frac{1}{2}\right)^2 \times$ Area of ΔABC.

Also from the similar triangles, $\angle ADF = \angle ABC$, so *DF* is parallel to *BC*. Therefore, since *AG* is perpendicular to *BC*, then *AG* is also perpendicular to *DE*, so *AE* is an altitude in ΔADF. Since ΔADF is isosceles, $DE = EF$.
Therefore,

$$\text{Area of } \Delta ADF = \tfrac{1}{2} \times \text{Area of } \Delta ADF$$
$$= \tfrac{1}{8} \times \text{Area of } \Delta ABC$$

Solution 2



Join *G* to *D* and *G* to *F*. This implies that we have four identical isosceles triangles: ΔADF, ΔGDF, ΔBDG, and ΔCFG. Each of these identical triangles has an equal area.
Thus,

$$\text{Area of } \Delta AEF = \tfrac{1}{2}\left(\tfrac{1}{4} \times \text{Area of } \Delta ABC\right)$$
$$= \tfrac{1}{8} \times \text{Area of } \Delta ABC$$

ANSWER: (E)

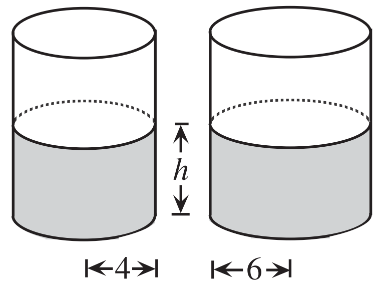
19. Let $x = 777\ 777\ 777\ 777\ 777$ and $y = 222\ 222\ 222\ 222\ 223$.

So the required quantity is $x^2 - y^2 = (x+y)(x-y)$.
Now $x + y = 1\,000\,000\,000\,000\,000$ and $x - y = 555\ 555\ 555\ 555\ 554$.
Thus, $x^2 - y^2 = 555\ 555\ 555\ 555\ 554\,000\,000\,000\,000\,000$, ie. the sum of the digits is $14 \times 5 + 4 = 74$.

ANSWER: (C)

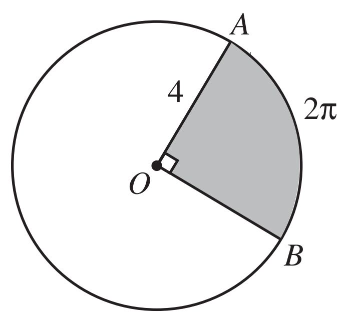
20. Let the final depth of the water be h.
The total initial volume of water is
$\pi(4\text{ m})^2(10\text{ m}) = 160\pi\text{ m}^3$.
When the depths of water are equal,

$$\pi(4\text{ m})^2 h + \pi(6\text{ m})^2 h = 160\pi\text{ m}^3$$
$$\left(52\pi\text{ m}^2\right)h = 160\pi\text{ m}^3$$
$$h = \frac{160}{52}\text{ m}$$
$$h = \frac{40}{13}\text{ m}$$



ANSWER: (E)

21. Let r be the radius of the circle.
Since $\angle AOB = 90^\circ$, then this sector is one quarter of the whole circle, so the circumference of the circle is $4 \times 2\pi = 8\pi$.
So $2\pi r = 8\pi$, or $r = 4$.
Thus the area of sector AOB is $\frac{1}{4}\pi r^2 = \frac{1}{4}\pi(16) = 4\pi$.



ANSWER: (A)

22. When we add up a number of consecutive integers, the sum of these integers will be equal to the number of integers being added times the average of the integers being added.
Let N be the number of consecutive integers being added, and let A be the average of the consecutive integers being added. Thus, $NA = 75$.
Before we determine the possibilities for N, we make the observation that N must be less than 12, since the sum of the first 12 positive integers is 78, and thus the sum of any 12 or more consecutive positive integers is at least 78.

Case 1: N is odd.

In this case, A (the average) is an integer (there is a "middle number" among the integers being added). Therefore, since $NA = 75$, N must be an odd positive factor of 75 which is bigger than 1 and less than 12, ie. N is one of 3 or 5. So there are two possibilities when N is odd, namely $24+25+26=75$ and $13+14+15+16+17=75$.

<u>Case 2</u>: N is even.
In this case A will be half-way between two integers.
Set $N = 2k$ and $A = \frac{2l+1}{2}$, where k and l are integers.
Then

$$2k\left(\frac{2l+1}{2}\right) = 75$$
$$k(2l+1) = 75$$

Thus k is a factor of 75, and so $N = 2k$ is 2 times a factor of 75, ie. N could be 2, 6 or 10. So the possibilities here are $37+38=75$, $10+11+12+13+14+15=75$, and $3+4+5+6+7+8+9+10+11+12=75$.

So there are 5 ways in total.

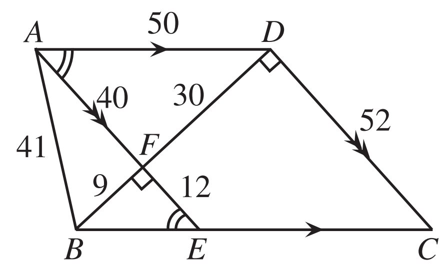
ANSWER: (E)

23. By Pythagoras in ΔAFB,

$$AF^2 + 9^2 = 41^2$$
$$AF = 40$$

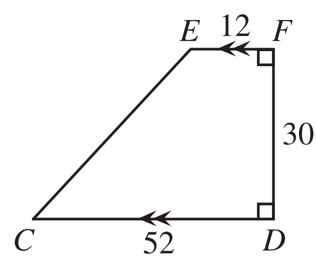
By Pythagoras in ΔAFD,

$$FD^2 + 40^2 = 50^2$$
$$FD = 30$$



Since AD is parallel to BC, $\angle FAD = \angle FEB$. Therefore, ΔAFD is similar to ΔEFB, so $\frac{AF}{FD} = \frac{EF}{FB}$ or $\frac{40}{30} = \frac{EF}{9}$ or $EF = 12$.
Since AE is perpendicular to BD and DC is perpendicular to BD, then AE is parallel to DC, so $AECD$ is a parallelogram. Thus, $DC = AE = 52$.
Now consider quadrilateral $FECD$.
This quadrilateral is a trapezoid, so

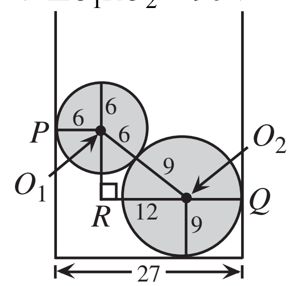
$$\text{Area of } FECD = \tfrac{1}{2}(EF+CD)(FD)$$
$$= \tfrac{1}{2}(12+52)(30)$$
$$= 960$$



ANSWER: (C)

24. We examine a vertical cross-section of the cylinder and the spheres that passes through the vertical axis of the cylinder and the centres of the spheres. (There is such a cross-section since the spheres will be pulled into this position by gravity.)
Let the centres of the spheres be O_1 and O_2, as shown.
Join the centres of the spheres to each other and to the respective points of tangency of the the spheres to the walls of the cylinder.
Then $O_1O_2 = 6+9=15$ (O_1O_2 passes through the point of tangency of the two spheres), $O_1P = 6$ and $O_2Q = 9$, using the radii of the spheres.
Next, draw ΔO_1RO_2 so that RO_1 is perpendicular to O_1P and $\angle O_1RO_2 = 90^\circ$.
Then looking at the width of the cylinder, since $PO_1 = 6$ and $O_2Q = 9$, we have

$$PO_1 + RO_2 + O_2Q = 27$$
$$RO_2 = 12$$



By Pythagoras in ΔO_1RO_2, we see that $O_1R = 9$.
Then the depth of the water will be

Radius of lower sphere $+ RO_1 +$ Radius of higher sphere $= 9+9+6 = 24$.

So

$$\text{Volume of water} = (\text{Volume of cylinder to height of } 24) - (\text{Volume of spheres})$$
$$= \pi\left(\tfrac{27}{2}\right)^2(24) - \tfrac{4}{3}\pi(6)^3 - \tfrac{4}{3}\pi(9)^3$$
$$= 4374\pi - 288\pi - 972\pi$$
$$= 3114\pi$$

Therefore, the volume of water required is 3114π cubic units.

ANSWER: (D)

25. Subtracting the first equation from the second,

$$k^2x - kx - 6 = 0$$
$$\left(k^2 - k\right)x = 6$$
$$\left[k(k-1)\right]x = 6$$

Since we want both k and x to be integers, then $k(k-1)$ is a factor of 6, ie. is equal to one of $\pm1, \pm2, \pm3, \pm6$. Now $k(k-1)$ is the product of two consecutive integers, so on this basis we can eliminate six of these eight possibilities to obtain

$k(k-1) = 2$ or $k(k-1) = 6$

which yields

$k^2 - k - 2 = 0$ or $k^2 - k - 6 = 0$

and so $k = 2, -1, 3, -2$.

We now make a table of values of k, x and y to check when y is also an integer. (We note that from the first equation, $y=\frac{1}{5}(kx+7)$.)

k	x	y
2	3	$\frac{13}{5}$
-1	3	$\frac{4}{5}$
3	1	2
-2	1	1

So there are two values of k for which the lines intersect at a lattice point.

ANSWER: (B)