

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2001 Solutions Fermat Contest (Grade 11)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING Awards

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Part A

1. If x + 2x + 3x + 4x = 5, then x equals (A) 10 (B) $\frac{1}{2}$ (C) $\frac{5}{4}$ (D) 2

Solution

$$x + 2x + 3x + 4x = 5$$

$$10x = 5$$

$$x = \frac{1}{2}$$

ANSWER: (B)

(E) $\frac{5}{9}$

2. If $x = \frac{1}{4}$, which of the following has the largest value?

(A) x (B) x^2 (C) $\frac{1}{2}x$ (D) $\frac{1}{x}$ (E) \sqrt{x}

Solution

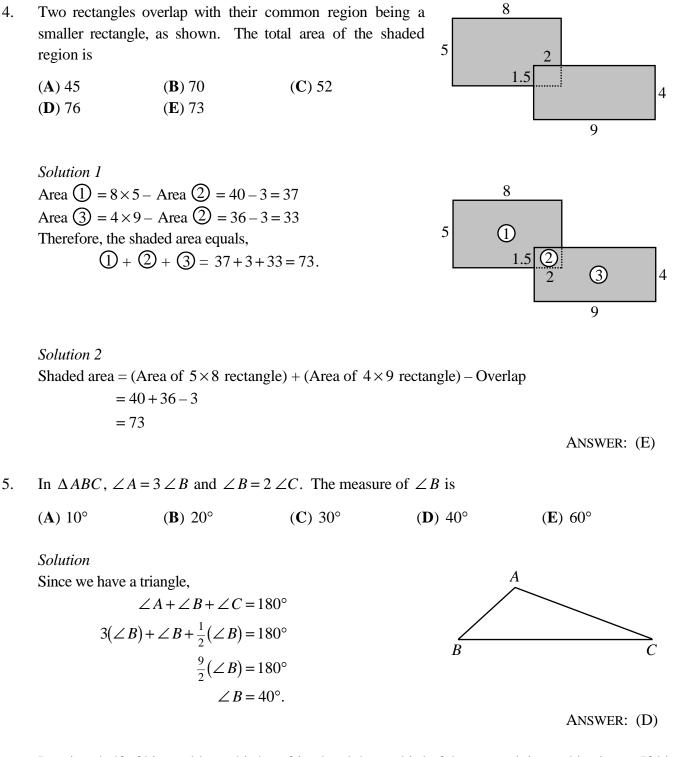
If we calculate the value of the given expressions, we get

- (A) $\frac{1}{4}$ (B) $\left(\frac{1}{4}\right)^2$ (C) $\frac{1}{2}\left(\frac{1}{4}\right)$ (D) $\frac{1}{\frac{1}{4}}$ (E) $\sqrt{\frac{1}{4}}$ $=\frac{1}{16}$ $=\frac{1}{8}$ $=1 \times 4$ $=\frac{1}{2}$ =4 ANSWER: (D)
- 3. In a school, 30 boys and 20 girls entered the Fermat competition. Certificates were awarded to 10% of the boys and 20% of the girls. Of the students who participated, the percentage that received certificates was

(A) 14 (B) 15 (C) 16 (D) 30 (E) 50

Solution

If 30 boys entered the Fermat competition and 10% of them received certificates, this implies that (0.1)(30) or 3 boys received certificates. Of the 20 girls who entered the competition (0.2)(20) or 4 girls received certificates. This implies that 7 students in total out of 50 received certificates. Thus 14% of the students in total received certificates. ANSWER: (A)

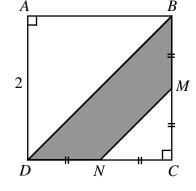


6. Pat gives half of his marbles to his best friend and then a third of those remaining to his sister. If his sister receives 9 marbles, then the number Pat keeps is

(A) 27 (B) 54 (C) 18 (D) 36 (E) 9

Solution Let *x* be the total number of marbles that Pat has initially. Then he gives $\frac{1}{2}x$ to his best friend, and $\frac{1}{3} \times \frac{1}{2}x = \frac{1}{6}x$ marbles to his sister. So $\frac{1}{6}x = 9$ x = 54. Pat keeps $x - \frac{1}{2}x - \frac{1}{6}x = \frac{1}{3}x = 18$ marbles.

- In the diagram, square ABCD has side length 2, with M the 7. midpoint of BC and N the midpoint of CD. The area of the shaded region BMND is
 - (A) 1 (B) $2\sqrt{2}$ (C) $\frac{4}{3}$ (D) $\frac{3}{2}$ (E) $4-\frac{3}{2}\sqrt{2}$



Solution

The area of ΔMNC is $\frac{1}{2}(1)(1) = \frac{1}{2}$. Since ΔBDC is half the square, it will have an area of 2. Since the shaded region has an area equal to that of ΔBDC minus the area of ΔMNC , its area will be $2 - \frac{1}{2} = \frac{3}{2}$. ANSWER: (D)

If $\sqrt{5+11-7} = \sqrt{5+11} - \sqrt{x}$, then the value of x is 8.

$$(A) 1 (B) 7 (C) -7 (D) 49 (E) 4$$

Solution

$$\sqrt{5+11-7} = \sqrt{5+11} - \sqrt{x}$$
$$\sqrt{9} = \sqrt{16} - \sqrt{x}$$
$$3 = 4 - \sqrt{x}$$
$$\sqrt{x} = 1$$
$$x = 1$$

ANSWER: (A)

- 9. A bag contains 20 candies: 4 chocolate, 6 mint and 10 butterscotch. Candies are removed randomly from the bag and eaten. What is the minimum number of candies that must be removed to be certain that at least two candies of each flavour have been eaten?
 - (**A**) 6 **(B)** 10 (**C**) 12 **(D)** 16 **(E)** 18

ANSWER: (C)

В

At most, 17 candies could be removed before the second chocolate candy is removed, that is all 10 butterscotch, all 6 mint, and 1 chocolate.

So we need to remove 18 candies to ensure that 2 of each flavour have been eaten.

ANSWER: (E)

10. When a positive integer N is divided by 60, the remainder is 49. When N is divided by 15, the remainder is

Solution

This problem can be done in a number of ways. The easiest way is to consider that if *N* is divided by 60 to achieve a remainder of 49, it must be a number of the form, 60k + 49, k = 0, 1, 2, ...

This implies that the smallest number to meet the requirements is 49 itself. If we divide 49 by 15 we get a remainder of 4. Or, if k = 1 in our formula then the next number to satisfy the requirements is 109 which when divided by 15 gives 4 as the remainder. ANSWER: (C)

Part B

11. The fourth root of 2001200120012001 (that is, $\frac{4}{2001200120012001}$) is closest to

(A) 2 001(B) 6 700(C) 21 000(D) 12 000(E) 2 100

Solution

2001200120012001 is roughly $2 \times 10^{15} = 2000 \times 10^{12}$. So the quantity desired is roughly $\sqrt[4]{2000 \times 10^{12}} = \sqrt[4]{2000} \times 10^3 \approx 7 \times 10^3$ which is closest to 6700. ANSWER: (B)

12. How many integer values of x satisfy
$$\frac{x-1}{3} < \frac{5}{7} < \frac{x+4}{5}$$
?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution

If we multiply all three fractions by 3(5)(7) we have,

$$(\mathfrak{S})(5)(7)\frac{(x-1)}{\mathfrak{S}} < (3)(5)(\mathfrak{F})\frac{5}{\mathfrak{F}} < (3)(\mathfrak{S})(7)\frac{x+4}{\mathfrak{S}}$$
$$35(x-1) < 75 < 21(x+4)$$

In order to satisfy this inequality then,

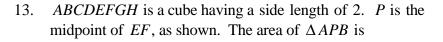
35(x-1) < 75 and 21(x+4) > 7535x-35 < 75 and 21x+84 > 75

$$35x < 110 \qquad 21x > -9$$

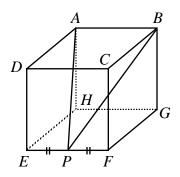
$$x < 3\frac{1}{7} \qquad x > -\frac{9}{21}$$

egers to satisfy both conditions are then in the set {0, 1, 2, 3}. ANSWER: (E)

The only integers to satisfy both conditions are then in the set $\{0, 1, 2, 3\}$.

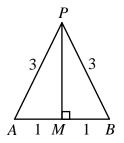


- (A) $\sqrt{8}$ (C) $\sqrt{32}$ **(B)** 3
- **(D)** $\sqrt{2}$ **(E)** 6



Solution

By symmetry, the lengths of AP and BP will be equal, and $AP = \sqrt{AD^2 + DE^2 + EP^2} = \sqrt{2^2 + 2^2 + 1^2} = 3.$ If M is the midpoint of AB, then PM is perpendicular to AB. By Pythagoras, $MP = \sqrt{3^2 - 1^2} = \sqrt{8}$. So the area of $\triangle APB$ is Area $=\frac{1}{2}(2)(\sqrt{8}) = \sqrt{8}$.





The last digit (that is, the units digit) of $(2002)^{2002}$ is 14.

(A) 4 **(B)** 2 (**C**) 8 **(D**) 0 (E) 6

Solution

The units digit of $(2002)^{2002}$ is the same as the units digit of 2^{2002} , since the first three digits of 2002 do not affect the units digit.

We write out the first few powers of 2 and check for the units digit.

n	1	2	3	4	5	6	7	8	9
2^n	2	4	8	16	32	64	128	256	512

From this table, we see that the units digits repeat every 4 powers. So the units digit of 2^{2000} will be 6, and thus the units digit of 2^{2002} (and so also of $(2002)^{2002}$) will be 4.

ANSWER: (A)

(A)
$$\frac{9}{2}$$
 (B) $\sqrt{17}$ (C) $\frac{17}{4}$
(D) $\frac{15}{4}$ (E) $\frac{\sqrt{17}}{2}$

$$(\mathbf{D}) \ \frac{15}{4} \qquad \qquad (\mathbf{E})$$

Solution

Let the centre be *C* and the radius *r*.

Then CQ is perpendicular to the y-axis, and has length r.

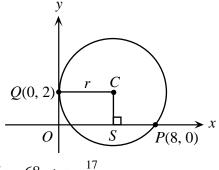
So the x-coordinate of C is r and the y-coordinate is 2, ie. C(r, 2).

Drop a perpendicular CS to the x-axis and consider right triangle CSP.

CP = r, CS = 2 and SP = 8 - r.

By Pythagoras, $r^2 = (8-r)^2 + 2^2 \implies r^2 = 64 - 16r + r^2 + 4 \implies 16r = 68 \implies r = \frac{17}{4}$.

у



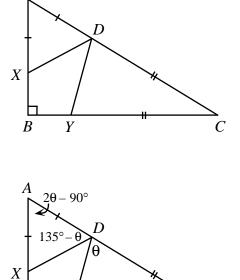
ANSWER: (C)

16. In right triangle ABC, AX = AD and CY = CD, as shown. The measure of $\angle XDY$ is

(A) 35°	(B) 40°	(C) 45°
(D) 50°	(E) not determined	by this information

Let $\angle DYC = \theta$. Then $\angle YDC = \theta$ ($\triangle YDC$ is isosceles).

Then $\angle YCD = 180^\circ - 2\theta$ (sum of angles in $\triangle YDC$). Then $\angle CAB = 2\theta - 90^\circ$ (sum of angles in $\triangle ABC$).



Y

В

Α

In $\triangle AXD$, $\angle AXD = \angle ADX$ so $2\theta - 90^\circ + 2\angle AXD = 180^\circ$. So $\angle AXD = 135^{\circ} - \theta$ and $\angle ADX = 135 - \theta$. But $\angle ADX + \angle XDY + \angle YDC = 180^{\circ}$.

Therefore, $135 - \theta + \angle XDY + \theta = 180^{\circ}$. Thus, $\angle XDY = 45^{\circ}$.

ANSWER: (C)

С

. 80° – 2θ

- 17. Three different numbers are chosen such that when each of the numbers is added to the average of the remaining two, the numbers 65, 69 and 76 result. The average of the three original numbers is
 - (A) 34 (B) 35 (C) 36 (D) 37 (E) 38

Let the three numbers be *a*, *b* and *c*.

We construct the first equation to be,

$$a + \frac{b+c}{2} = 65.$$

Or, 2a + b + c = 130.

Similarly we construct the two other equations to be,

$$a+2b+c=138$$

and $a+b+2c=152$.

If we add the three equations we obtain,

$$4a + 4b + 4c = 420.$$

The average is $\frac{4(a+b+c)}{12} = \frac{420}{12}.$
Or, $\frac{a+b+c}{3} = 35.$

18. In the diagram, the two smaller circles have equal radii. Each of the three circles is tangent to the other two circles, and each is also tangent to sides of the rectangle. If the width of the rectangle is 4, then its length is

(A)
$$2 + \sqrt{8}$$
 (B) $3 + \sqrt{8}$ (C) $3 + \sqrt{10}$
(D) $\sqrt{32}$ (E) $4 + \sqrt{3}$

Solution

Let the radius of the larger circle be *R*.

Let the radii of the smaller circle be *r*.

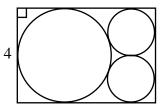
From the diagram, 2R = 4 so R = 2 and 4r = 4 so r = 1, as the radii are perpendicular to the sides of the rectangle since the circles are tangent to the sides of the rectangle.

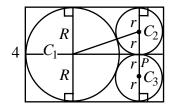
Join C_1 to P and extend until it hits both vertical sides of the rectangle.

Therefore the length of the rectangle is $R + C_1P + r = 3 + C_1P$.

Now $C_2P \perp C_1P$ and C_2C_1 goes through the point of tangency between the larger circle and the upper circle, so $C_1C_2 = R + r = 3$.

By Pythagoras
$$3^2 = (C_1 P)^2 + 1^2 \Rightarrow (C_1 P)^2 = 8 \Rightarrow C_1 P = \sqrt{8}$$
.
Thus the length is $3 + \sqrt{8}$.





ANSWER: (B)

ANSWER: (B)

19. Cindy leaves school at the same time every day. If she cycles at 20 km/h, she arrives home at 4:30 in the afternoon. If she cycles at 10 km/h, she arrives home at 5:15 in the afternoon. At what speed, in km/h, must she travel to arrive home at 5:00 in the afternoon?

(A)
$$16\frac{2}{3}$$
 (B) 15 (C) $13\frac{1}{3}$ (D) 12 (E) $18\frac{3}{4}$

Solution

Since the distance from Cindy's home to school is unknown, represent this distance by d, in kilometres. We will consider the problem in two separate cases, the first in which she travels at 20 km/h and the second when she travels at 10 km/h.

Distance travelled at 20 km/h = Distance travelled at 10 km/h

Let the time that Cindy takes travelling home at 20 km/h be t hours.

If Cindy arrives home $\frac{3}{4}$ h later when travelling at 10 km/h, then the length of time travelling is $\left(t + \frac{3}{4}\right)$ hours. The previous equation becomes

$$20t = 10\left(t + \frac{3}{4}\right)$$

$$20t = 10t + \frac{30}{4}$$

$$10t = \frac{15}{2}$$

$$t = \frac{15}{20} \text{ or } \frac{3}{4}.$$

Therefore the distance from school to home is $d = 20 \times \frac{3}{4}$, or d = 15 km.

If Cindy arrives home at 5:00 in the afternoon, she would have travelled home in $\frac{3}{4} + \frac{1}{2} = \frac{5}{4}$ hours over a distance of 15 kilometres.

Therefore,
$$s = \frac{d}{t} = \frac{15}{\frac{5}{4}} = 15 \times \frac{4}{5} = 12 \text{ km/h}$$

Therefore, Cindy would have had to travel at 12 km/h to arrive home at 5:00 p.m.

ANSWER: (D)

20. Point *P* is on the line y = 5x + 3. The coordinates of point *Q* are (3, -2). If *M* is the midpoint of *PQ*, then *M* must lie on the line

(A)
$$y = \frac{5}{2}x - \frac{7}{2}$$
 (B) $y = 5x + 1$ (C) $y = -\frac{1}{5}x - \frac{7}{5}$ (D) $y = \frac{5}{2}x + \frac{1}{2}$ (E) $y = 5x - 7$

We start by drawing a diagram and labelling the intercepts.

Solution 1

Since the point *P* is on the line y = 5x + 3, select P(0, 3) as a point on this line.

The midpoint of *PQ* is $M\left(\frac{3+0}{2}, \frac{-2+3}{2}\right) = M\left(\frac{3}{2}, \frac{1}{2}\right)$.

The required line must contain *M* and be midway between the given point and y = 5x + 3. The only possible line meeting this requirement is the line containing $M\left(\frac{3}{2}, \frac{1}{2}\right)$ and which has a slope of 5. The required line will this have as its equation

$$y - \frac{1}{2} = 5\left(x - \frac{3}{2}\right)$$

or, $y = 5x - 7$.

$\begin{array}{c} y \\ P(0, 3) \\ (-\frac{3}{5}, 0) \\ y = 5x + 3 \end{array}$

Solution 2

Let a general point on the line y = 5x + 3 be represented by (a, 5a + 3). Also, let a point on the required line be M(x, y). Since M(x, y) is the midpoint of PQ then

(1)
$$x = \frac{a+3}{2}$$
 and (2) $y = \frac{(5a+3)+(-2)}{2}$
 $y = \frac{5a+1}{2}$

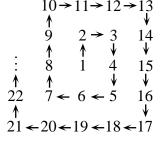
Solving (1) for *a*, we have a = 2x - 3 and solving (2) for *a*, we have $\frac{2y - 1}{5} = a$.

Equating gives, $2x - 3 = \frac{2y - 1}{5}$ 10x - 15 = 2y - 1or, y = 5x - 7. ANSWER: (E)

Part C

21. A spiral of numbers is created, as shown, starting with 1. If the pattern of the spiral continues, in what configuration will the numbers 399, 400 and 401 appear?

(A)
$$399 \rightarrow 400 \rightarrow 401$$
 (B) $401 \leftarrow 400 \leftarrow 399$



(C)	401	(D)	399	(E)	$400 \rightarrow 401$
	↑ 400		¥ 400		↑ 399
	400		400		399
	399		401		

We notice the following configurations:

	2 1	3 4 5		
and	10 9	11 2	12 3	13 14
	8	1	4	15
	7	6	5	(16)
				17

And continuing the pattern of the spiral

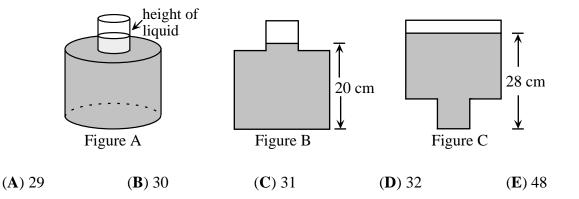
26	27	28	29	30	31
25	10	11	12	13	32
24	9	2	3	14	33
23	8	1	4	15	34
22	7	6	5	11	35
21	20	19	18	17	36
					37

So we see that when the pattern continues, all even perfect squares will occur in the configuration $(2k)^2 - 1$

 \downarrow $(2k)^{2}$ \downarrow $(2k)^{2} + 1$ so we have 399 as $400 = (20)^{2}$. \downarrow 400 \downarrow 401

ANSWER: (D)

22. A sealed bottle, which contains water, has been constructed by attaching a cylinder of radius 1 cm to a cylinder of radius 3 cm, as shown in Figure A. When the bottle is right side up, the height of the water inside is 20 cm, as shown in the cross-section of the bottle in Figure B. When the bottle is upside down, the height of the liquid is 28 cm, as shown in Figure C. What is the total height, in cm, of the bottle?



Solution

We'll start by representing the height of the large cylinder as h_1 and the height of the small cylinder as h_2 . For simplicity, we'll let $x = h_1 + h_2$.

If the bottom cylinder is completely filled and the top cylinder is only partially filled the top cylinder will have a cylindrical space that is not filled. This cylindrical space will have a height equal to x - 20 and a volume equal to, $\pi(1)^2(x - 20)$.

Similarly, if we turn the cylinder upside down there will be a cylindrical space unfilled that will have a height equal to x - 28 and a volume equal to, $\pi(3)^2(x - 28)$.

Since these two unoccupied spaces must be equal, we then have,

$$\pi(1)^{2}(x-20) = \pi(3)^{2}(x-28)$$
$$x-20 = 9x-252$$
$$8x = 272$$
$$x = 29.$$

Therefore, the total height is 29.

23. A sequence $t_1, t_2, ..., t_n, ...$ is defined as follows:

$$t_1 = 14$$

 $t_k = 24 - 5t_{k-1}$, for each $k \ge 2$.

For every positive integer *n*, t_n can be expressed as $t_n = p \cdot q^n + r$, where *p*, *q* and *r* are constants. The value of p + q + r is

(A) -5 (B) -3 (C) 3 (D) 17 (E) 31

ANSWER: (A)

Solution 1 Since $t_n = p \cdot q^n + r$ for every $n \ge 1$, then $t_1 = pq + r$ $t_2 = pq^2 + r$ $t_3 = pq^3 + r.$ However, $t_1 = 14$, $t_2 = 24 - 5(t_1) = 24 - 5(14) = -46$, and $t_3 = 24 - 5t_2 = 24 - 5(-46) = 254$. So pq + r = 14 (1) $pq^2 + r = -46$ (2) $pq^3 + r = 254$ (3) Subtracting (2) – (1) yields $pq^2 - pq = pq(q-1) = -60$ (4)Subtracting (3) – (2) yields $pq^3 - pq^2 = pq^2(q-1) = 300$ (5) Dividing (5) by (4) gives q = -5. Substituting back into (1) and (2)-5p + r = 14(1)25p + r = -46 (2) Adding $5 \times (1)$ to (2) yields 6r = 24 so r = 4 and thus p = -2. Therefore $t_n = -2(-5)^n + 4$. So p + q + r = -2 - 5 + 4 = -3.

Solution 2 Substituting $t_n = p \cdot q^n + r$ and $t_{n-1} = p \cdot q^{n-1} + r$ into the difference equation $t_n = 24 - 5t_{n-1}$, we get $p \cdot q^n + r = 24 - 5(p \cdot q^{n-1} + r)$ $p \cdot q^n + 5pq^{n-1} = 24 - 5r - r$ $p \cdot q^{n-1}(q+5) = 24 - 6r$.

Now the right side is independent of *n*, so the left side must be as well, thus q+5=0 or q=-5. (Clearly, $p \neq 0$ or else t_n is constant).

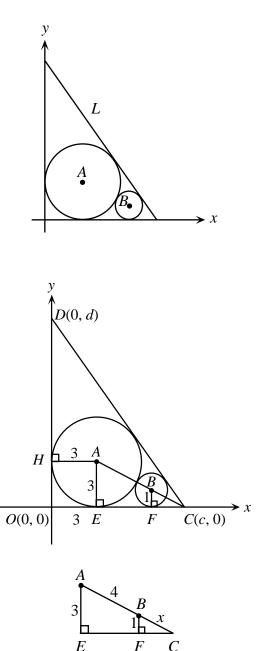
So
$$24 - 6r = 0 \Rightarrow r = 4$$
.
Therefore $t_n = p(-5)^n + 4$, so $t_1 = 14 = -5p + 4 \Rightarrow p = -2$.
Therefore $p + q + r = -3$.
ANSWER: (B)

24. The circle with centre A has radius 3 and is tangent to both the positive x-axis and positive y-axis, as shown. Also, the circle with centre B has radius 1 and is tangent to both the positive x-axis and the circle with centre A. The line L is tangent to both circles. The y-intercept of line L is

(A)
$$3+6\sqrt{3}$$
 (B) $10+3\sqrt{2}$ (C) $8\sqrt{3}$
(D) $10+2\sqrt{3}$ (E) $9+3\sqrt{3}$



We start by drawing a line from point *C* that will pass through *A* and *B*. From *A* and *B*, we drop perpendiculars to the points of tangency on the *x*-axis and label these points as *E* and *F* as shown. We also drop a perpendicular from *A* to the *y*-axis which makes AH = AE = 3.



Extracting ΔCAE from the diagram and labelling with the given information we would have the following noted in the diagram.

If we represent the distance from C to B as x and recognize that $\triangle CBF$ is similar to $\triangle CAE$,

$$\frac{x}{1} = \frac{x+4}{3}$$

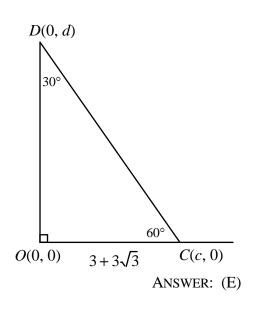
x = 2.
In $\triangle CBF$, $FC^2 = 2^2 - 1^2 = 3$
 $FC = \sqrt{3}$, $(FC > 0)$.

This implies that $\angle BCF = 30^{\circ}$ and $\angle OCD = 60^{\circ}$. Therefore $EF = 2\sqrt{3}$, from similar triangles again.

This now gives us the diagram shown.

Thus,
$$d = \sqrt{3}(3+3\sqrt{3})$$

= $3\sqrt{3}+9$.



25. A square array of dots with 10 rows and 10 columns is given. Each dot is coloured either blue or red. Whenever two dots of the same colour are adjacent in the same row or column, they are joined by a line segment of the same colour as the dots. If they are adjacent but of different colours, they are then joined by a green line segment. In total, there are 52 red dots. There are 2 red dots at corners with an additional 16 red dots on the edges of the array. The remainder of the red dots are inside the array. There are 98 green line segments. The number of blue line segments is

•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•

(A) 36
(B) 37
(C) 38
(D) 39
(E) 40

Solution

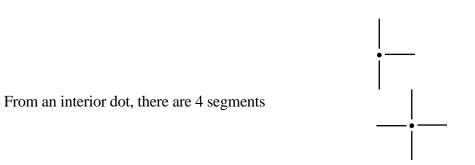
First, we note that there are 9 line segments in each row and in each column, so there are 9(10) + 9(10) = 180 line segments in total.

Let *B* be the number of blue segments and *R* the number of red segments. Then B + R + 98 = 180, so B + R = 82, as there are 98 green line segments.

Coming out of a red dot, there can only be a green line segment or a red line segment. We count the *total number* of line segments starting from red dots. Note that in this total, the green segments are counted once and the red segments twice, as the red segments have both ends at red dots.

From a corner dot, there are 2 segments

From an edge dot (not on corner), there are 3 segments



So the total number of segments coming from red dots is

2(2) + 3(16) + 4(34) = 188

and so, since 98 segments from red dots are accounted for by green segments the remaining 188-98=90 segments from red dots are accounted for by red segments, for a total of 45 red segments, or R = 45.

Therefore B = 82 - R = 37.

ANSWER: (B)