Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2000 Solutions
Fermat Contest (Grade 11)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

Part A:

1. The sum $29+12+23$ is equal to

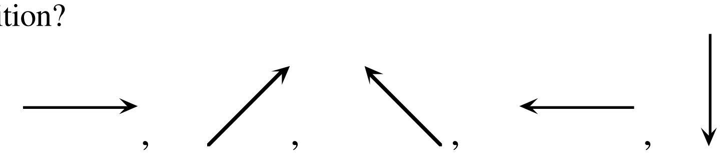
(**A**) 6^2 (**B**) 4^4 (**C**) 8^8 (**D**) 64^0 (**E**) 2^6

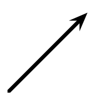
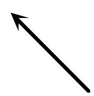
Solution

$$\begin{aligned} 29+12+23 &= 64 \\ &= 2^6 \end{aligned}$$

ANSWER: (E)

2. If the following sequence of five arrows repeats itself continuously, what arrow will be in the 48th position?



(**A**)  (**B**)  (**C**)  (**D**)  (**E**) 

Solution
Since this sequence repeats itself, once it has completed nine cycles it will be the same as starting at the beginning. Thus the 48th arrow will be the same as the third one.

ANSWER: (C)

3. A farmer has 7 cows, 8 sheep and 6 goats. How many more goats should be bought so that half of her animals will be goats?

(**A**) 18 (**B**) 15 (**C**) 21 (**D**) 9 (**E**) 6

Solution 1
If the cows and sheep were themselves goats we would have 15 goats. This means that she would need nine extra goats.

Solution 2
Let the number of goats added be x.
Therefore, $\frac{6+x}{21+x}=\frac{1}{2}$
Cross multiplying gives,
$$2(6+x)=21+x$$
$$12+2x=21+x$$
$$x=9.$$
As in solution 1, she would add 9 goats.

ANSWER: (D)

4. The square of 9 is divided by the cube root of 125. What is the remainder?

(**A**) 6 (**B**) 3 (**C**) 16 (**D**) 2 (**E**) 1

Solution
The square of 9 is 81 and the cube root of 125 is 5. When 81 is divided by 5, the quotient is 16 and the remainder is 1.
Since $81=5\times16+1$, the remainder is 1.

ANSWER: (E)

5. The product of 2, 3, 5, and y is equal to its sum. What is the value of y?

(**A**) $\frac{1}{3}$ (**B**) $\frac{10}{31}$ (**C**) $\frac{10}{29}$ (**D**) $\frac{3}{10}$ (**E**) $\frac{10}{3}$

Solution
Since
$$(2)(3)(5)(y)=2+3+5+y.$$
$$30y=10+y$$
$$29y=10$$
and
$$y=\frac{10}{29}.$$

ANSWER: (C)

6. A student uses a calculator to find an answer but instead of pressing the $\boxed{x^2}$ key presses the $\boxed{\sqrt{x}}$ key by mistake. The student's answer was 9. What should the answer have been?

(**A**) 243 (**B**) 81 (**C**) 729 (**D**) 3 (**E**) 6561

Solution 1
Since $\sqrt{x}=9$, $x=81$.
Thus, $x^2=81^2=6561$.

Solution 2
Since the square root of the number entered is 9, the number must have been 81. The answer desired is $81^2 = 6561$.

ANSWER: (E)

7. The sum of the arithmetic series $(-300) + (-297) + (-294) + \ldots + 306 + 309$ is

(**A**) 309 (**B**) 927 (**C**) 615 (**D**) 918 (**E**) 18

Solution
The given series is
$(-300) + (-297) + (-294) + \ldots + (-3) + 0 + 3 + \ldots + 294 + 297 + 300 + 303 + 306 + 309.$
The sum of the terms from –300 to 300 is 0. The sum of all the terms is thus
$303 + 306 + 309 = 918$.

ANSWER: (D)

8. In a school referendum, $\frac{3}{5}$ of a student body voted 'yes' and 28% voted 'no'. If there were no spoiled ballots, what percentage of the students did not vote?

(**A**) 72% (**B**) 40% (**C**) 32% (**D**) 12% (**E**) 88%

Solution
Since $\frac{3}{5}$ or 60% of the students voted 'yes' and 28% voted 'no', 88% of the student body voted. Hence, $100\% - 88\%$ or 12%, of the students did not vote.

ANSWER: (D)

9. The numbers 6, 14, x, 17, 9, y, 10 have a mean of 13. What is the value of $x + y$?

(**A**) 20 (**B**) 21 (**C**) 23 (**D**) 25 (**E**) 35

Solution
If the 7 numbers have a mean of 13, this implies that these numbers would have a sum of 7×13 or 91. We now can calculate $x + y$ since, $6 + 14 + x + 17 + 9 + y + 10 = 91$ or, $x + y + 56 = 91$
Therefore, $x + y = 35$.

ANSWER: (E)

10. If $x(x(x+1)+2)+3 = x^3 + x^2 + x - 6$ then x is equal to

(**A**) 11 (**B**) -9 (**C**) -4 or 3 (**D**) –1 or 0 (**E**) -2

Solution

$$x(x(x+1)+2)+3 = x(x^2+x+2)+3$$
$$= x^3+x^2+2x+3$$

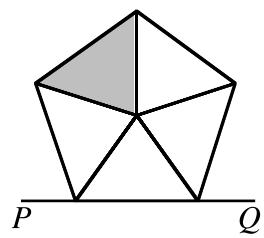
Since $x^3+x^2+2x+3 = x^3+x^2+x-6$

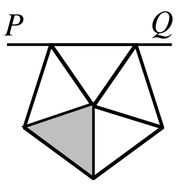
$$2x+3 = x-6$$
$$x = -9$$

ANSWER: (B)

Part B:

11. When the regular pentagon is reflected in the line PQ, and then rotated *clockwise* 144° about the centre of the pentagon, its position is



(**A**)


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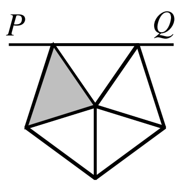
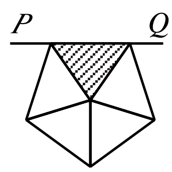


(**C**)


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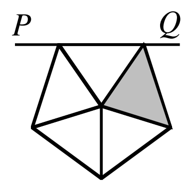
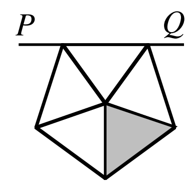


(**E**)


Solution

After the reflection, the new pentagon is

Since each of the five central angles equals 72°, a clockwise rotation of $144^\circ\left(2\times 72^\circ\right)$ about its centre will place the pentagon in the shown position.

ANSWER: (C)

12. If the expression $15^6 \times 28^5 \times 55^7$ was evaluated, it would end with a string of consecutive zeros. How many zeros are in this string?

(**A**) 10 (**B**) 18 (**C**) 26 (**D**) 13 (**E**) 5

Solution

A zero at the end of a number results from the product of 2 and 5. The number of zeros at the end of a number equals the number of product pairs of 2 and 5 that can be formed from the prime factorization of that number.

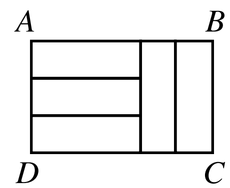
Since $15^6 \times 28^5 \times 55^7 = (3.5)^6\left(2^2.7\right)^5(5.11)^7$

$= 3^6.5^6.2^{10}.7^5.5^7.11^7$
$= 3^6.5^3.7^5.11^7.10^{10}$

There will be ten zeros in the string at the end of the number.

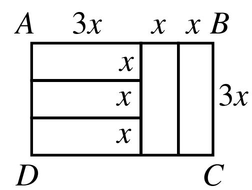
ANSWER: (A)

13. Rectangle $ABCD$ is divided into five congruent rectangles as shown. The ratio $AB:BC$ is



(**A**) 3:2 (**B**) 2:1 (**C**) 5:2
(**D**) 5:3 (**E**) 4:3

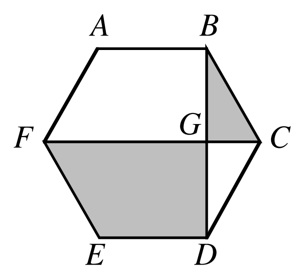
Solution



If we let the width of each rectangle be x units then the length of each rectangle is $3x$ units. (This is illustrated in the diagram.) The length, AB, is now $3x + x + x$ or $5x$ units and $BC = 3x$. Thus $AB:BC = 5x:3x$
$= 5:3, \ x \neq 0.$

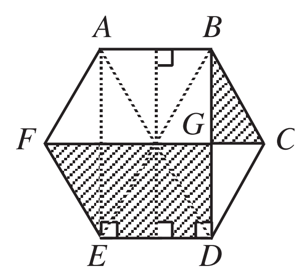
ANSWER: (D)

14. In the regular hexagon $ABCDEF$, two of the diagonals, FC and BD, intersect at G. The ratio of the area of quadrilateral $FEDG$ to the area of ΔBCG is



(**A**) $3\sqrt{3}:1$ (**B**) 4:1 (**C**) 6:1
(**D**) $2\sqrt{3}:1$ (**E**) 5:1

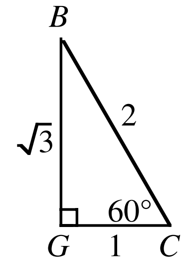
Solution 1



Join E to B and D to A as shown. Also join E to A and draw a line parallel to AE through the point of intersection of BE and AD. Quadrilateral $FEDG$ is now made up of five triangles each of which has the same area as ΔBCG. The required ratio is 5:1.

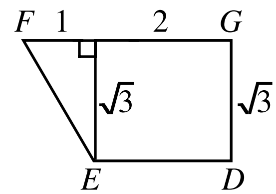
Solution 2

For convenience, assume that each side of the hexagon has a length of 2 units. Each angle in the hexagon equals 120° so $\angle BCG = \frac{1}{2}(120^\circ) = 60^\circ$. Now label ΔBCG as shown. Using the standard ratios for a $30^\circ - 60^\circ - 90^\circ$ triangle we have $BG = \sqrt{3}$ and $GC = 1$.



The area of $\Delta BCG = \frac{1}{2}(1)\sqrt{3} = \frac{\sqrt{3}}{2}$. Dividing the quadrilateral $FGDE$ as illustrated, it will have an area of

$$2\left(\sqrt{3}\right) + \frac{1}{2}(1)\left(\sqrt{3}\right) = \frac{5\sqrt{3}}{2}.$$



The required ratio is $\frac{5\sqrt{3}}{2} : \frac{\sqrt{3}}{2}$ or 5:1, as in solution 1.

ANSWER: (E)

15. In a sequence, every term after the second term is twice the sum of the two preceding terms. The seventh term of the sequence is 8, and the ninth term is 24. What is the eleventh term of the sequence?

(**A**) 160 (**B**) 304 (**C**) 28 (**D**) 56 (**E**) 64

Solution

Let the seventh through eleventh terms of the sequence be t_7, t_8, t_9, t_{10}, and t_{11}.

Since
$$t_9 = 2(t_7 + t_8),$$
$$24 = 2(8 + t_8)$$
$$12 = 8 + t_8$$
$$t_8 = 4.$$
Hence
$$t_{10} = 2(t_8 + t_9) = 2(4 + 24) = 56$$
and
$$t_{11} = 2(24 + 56)$$
$$= 160.$$

ANSWER: (A)

16. The digits 2, 2, 3, and 5 are randomly arranged to form a four digit number. What is the probability that the sum of the first and last digits is even?

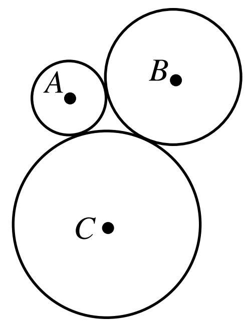
(**A**) $\frac{1}{4}$ (**B**) $\frac{1}{3}$ (**C**) $\frac{1}{6}$ (**D**) $\frac{1}{2}$ (**E**) $\frac{2}{3}$

Solution

The numbers that can be formed from the digits 2, 2, 3, and 5, in ascending order, are 2235, 2253, 2325, 2352, 2523, 2532, 3225, 3252, 3522, 5223, 5232, and 5322. These are twelve possibilities. For the sum of two digits to be even, both must be even or both must be odd. From the above list, the numbers for which the sum of the first and last digits is even, are 2352, 2532, 3225, and 5223. These are four possibilities. Thus, the probability of getting one of these numbers is $\frac{4}{12}$ or $\frac{1}{3}$.

ANSWER: (B)

17. Three circles have centres A, B and C with radii 2, 4 and 6 respectively. The circles are tangent to each other as shown. Triangle ABC has



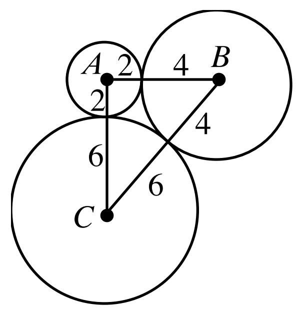
(**A**) $\angle A$ obtuse (**B**) $\angle B = 90°$ (**C**) $\angle A = 90°$

(**D**) all angles acute (**E**) $\angle B = \angle C$

Solution

Since the circles are mutually tangent, the lines joining their centres pass through the points of tangency. Thus, the sides of ΔABC have lengths 6, 8 and 10 if we write in the radii as shown.

Since $10^2 = 6^2 + 8^2$, the triangle is right-angled at A.



ANSWER: (C)

18. If $P = 3^{2000} + 3^{-2000}$ and $Q = 3^{2000} - 3^{-2000}$ then the value of $P^2 - Q^2$ is

(**A**) 3^{4000} (**B**) 2×3^{-4000} (**C**) 0 (**D**) 2×3^{4000} (**E**) 4

Solution 1

$$P^2 - Q^2 = (P+Q)(P-Q) = \left[\left(3^{2000} + 3^{-2000}\right) + \left(3^{2000} - 3^{-2000}\right)\right]\left[\left(3^{2000} + 3^{-2000}\right) - \left(3^{2000} - 3^{-2000}\right)\right]$$

$$= \left(2.3^{2000}\right)\left(2.3^{-2000}\right)$$

$$= 4.3^{\circ}$$

$$= 4$$

Solution 2

$$P^2 - Q^2 = \left(3^{2000} + 3^{-2000}\right)^2 - \left(3^{2000} - 3^{-2000}\right)^2$$

$$= 3^{4000} + 2.3^{2000}.3^{-2000} + 3^{-4000} - 3^{4000} + 2.3^{2000}.3^{-2000} - 3^{-4000}$$
$$= 3^{4000} + 2.3^{\circ} + 3^{-4000} - 3^{4000} + 2.3^{\circ} - 3^{-4000}$$
$$= 4$$

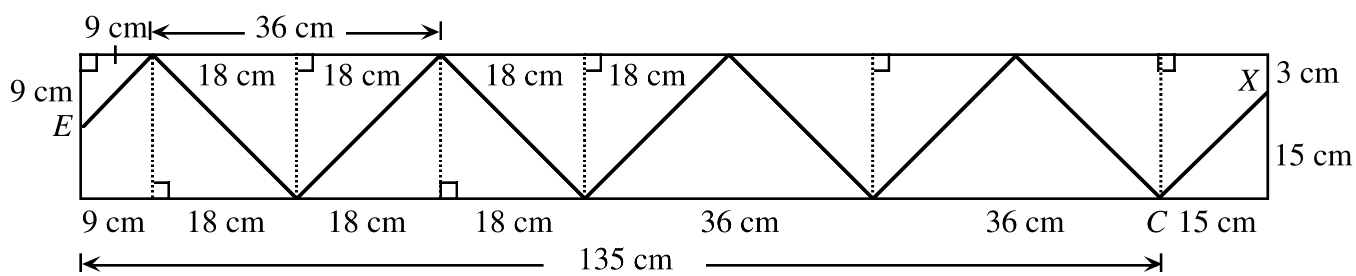
ANSWER: (E)

19. An ant walks inside a 18 cm by 150 cm rectangle. The ant's path follows straight lines which always make angles of 45° to the sides of the rectangle. The ant starts from a point X on one of the shorter sides. The first time the ant reaches the opposite side, it arrives at the mid-point. What is the distance, in centimetres, from X to the nearest corner of the rectangle?

(**A**) 3 (**B**) 4 (**C**) 6 (**D**) 8 (**E**) 9

Solution

If we took a movie of the ant's path and then played it backwards, the ant would now start at the point E and would then end up at point X. Since the ant now 'starts' at a point nine cm from the corner, the 'first' part of his journey is from E to B. This amounts to nine cm along the length of the rectangle since ΔBAE is an isosceles right-angled triangle. This process continues as illustrated, until the ant reaches point C. By the time the ant has reached C, it has travelled $9 + 18 + 3 \times 36$ or 135 cm along the length of the rectangle. To travel from C to X, the ant must travel 15 cm along the length of the rectangle which puts the ant 3 cm from the closest vertex.



ANSWER: (A)

20. Given $a + 2b + 3c + 4d + 5e = k$ and $5a = 4b = 3c = 2d = e$ find the smallest positive integer value for k so that a, b, c, d, and e are all positive integers.

(**A**) 87 (**B**) 522 (**C**) 10 (**D**) 120 (**E**) 60

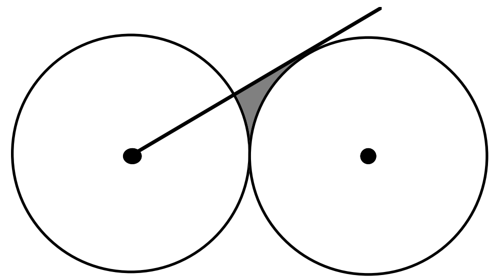
Solution

From the equalities $5a = 4b = 3c = 2d = e$, we conclude that e is the largest and a is the smallest of the integers. Since e is divisible by 5, 4, 3, and 2, the smallest possible value for e is 60. The corresponding values for a, b, c, and d, are 12, 15, 20, and 30, respectively. Thus, the smallest positive integer value for k is $12 + 2(15) + 3(20) + 4(30) + 5(60) = 522$.

ANSWER: (B)

Part C:

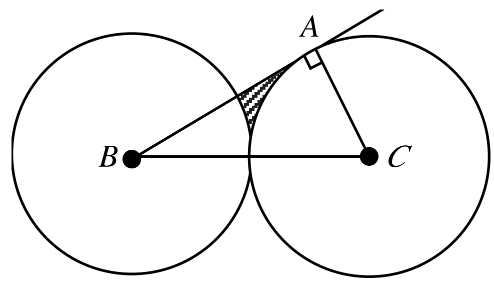
21. Two circles of radius 10 are tangent to each other. A tangent is drawn from the centre of one of the circles to the second circle. To the nearest integer, what is the area of the shaded region?



(**A**) 6 (**B**) 7 (**C**) 8
(**D**) 9 (**E**) 10

Solution

Let the centres of the circles be B and C and the point of contact of the tangent line from B to the circle with centre C be A. Thus, $\angle CAB = 90^\circ$ using properties of tangents to a circle.



Since the sides of right-angled triangle ABC are in the ratio of $1:2:\sqrt{3}$, $\angle B = 30^\circ$ and $\angle C = 60^\circ$. The area of the shaded region is equal to the area of ΔABC minus the area of the two sectors of the circles.

The area of the two sectors is equivalent to $\frac{1}{4}$ the area of a circle with radius 10.

The area of the shaded region $= \frac{1}{2}(10)\left(10\sqrt{3}\right) - \frac{1}{4}\pi(10)^2$

$= 50\sqrt{3} - 25\pi$

≈ 8.063

To the nearest integer, the area is 8. ANSWER: (C)

22. The left most digit of an integer of length 2000 digits is 3. In this integer, any two consecutive digits must be divisible by 17 or 23. The 2000th digit may be either 'a' or 'b'. What is the value of $a + b$?

(**A**) 3 (**B**) 7 (**C**) 4 (**D**) 10 (**E**) 17

Solution

We start by noting that the two-digit multiples of 17 are 17, 34, 51, 68, and 85. Similarly we note that the two-digit multiples of 23 are 23, 46, 69, and 92. The first digit is 3 and since the only two-digit number in the two lists starting with 3 is 34, the second digit is 4. Similarly the third digit must be 6. The fourth digit, however, can be either 8 or 9. Let's consider this in two cases.

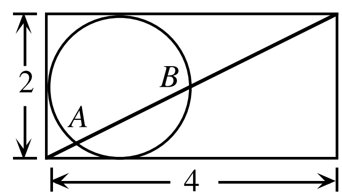
Case 1
If the fourth digit is 8, the number would be 3468517 and would stop here since there isn't a number in the two lists starting with 7.

Case 2
If the fourth digit is 9, the number would be 34692 34692 34 ... and the five digits '34692' would continue repeating indefinitely as long as we choose 9 to follow 6.

If we consider a 2000 digit number, its first 1995 digits must contain 399 groups of '34692'. The last groups of five digits could be either 34692 or 34685 which means that the 2000th digit may be either 2 or 5 so that $a+b=2+5=7$.

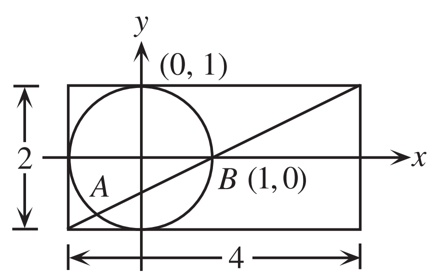
ANSWER: (B)

23. A circle is tangent to three sides of a rectangle having side lengths 2 and 4 as shown. A diagonal of the rectangle intersects the circle at points A and B. The length of AB is



(A) $\sqrt{5}$ (B) $\frac{4\sqrt{5}}{5}$ (C) $\sqrt{5}-\frac{1}{5}$ (D) $\sqrt{5}-\frac{1}{6}$ (E) $\frac{5\sqrt{5}}{6}$

Solution 1
Of the many ways to solve this problem perhaps the easiest is to use a Cartesian co-ordinate system. There are two choices for the origin, the centre of the circle and the bottom left vertex of the rectangle. The first solution presented uses the centre of the circle as the origin where the axes are lines drawn parallel to the sides of the rectangle through $(0,0)$. The equation of the circle is now $x^2+y^2=1$ and the equation of the line containing the diagonal is $y+1=\frac{1}{2}(x+1)$ or $x-2y=1$.



Equation of Line: $y+1=\frac{1}{2}(x+1) \Leftrightarrow x-2y=1$
Equation of Circle: $x^2+y^2=1$

Solving to find the intersection points of the line and circle gives, $(2y+1)^2+y^2=1$

$$5y^2+4y=0$$
$$y(5y+4)=0$$
$$y=0 \text{ or } y=-\frac{4}{5}$$

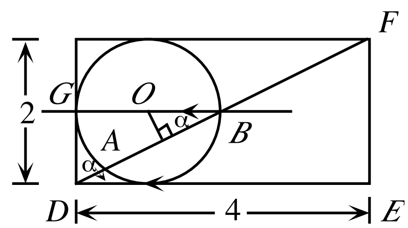
If we substitute these values of y into $x-2y=1$ we find,
The points of intersection are $A\left(\frac{-3}{5},-\frac{4}{5}\right)$ and $B(1,0)$.

The length of AB is $\sqrt{\left(1+\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}=\sqrt{\frac{80}{25}}=\frac{4\sqrt{5}}{5}$.

Note: The other suggested choice for the origin would involve using the equation of the line $y=\frac{1}{2}x$ and the circle $(x-1)^2+(y-1)^2=1$.

Solution 2

The diameter, BG, is parallel to the sides of the rectangle and has its endpoint at B which is the centre of the rectangle.



Draw $OC \perp AB$ where O is the centre of the circle and by chord properties of a circle, $CB=\frac{1}{2}AB$.

Since $OB \parallel DE$, $\angle BDE=\angle CBO=\alpha$.

Thus, ΔOBC is similar to ΔFDE.

Since $DF=2\sqrt{5}$, $\frac{1}{2\sqrt{5}}=\frac{CB}{4}$ and $CB=\frac{2}{5}\sqrt{5}$.

The length of AB is $2\left(\frac{2}{5}\sqrt{5}\right)$ or $\frac{4}{5}\sqrt{5}$.

ANSWER: (B)

24. For the system of equations $x^2+x^2y^2+x^2y^4=525$ and $x+xy+xy^2=35$, the sum of the real y values that satisfy the equations is

(**A**) 20 (**B**) 2 (**C**) 5 (**D**) $\frac{55}{2}$ (**E**) $\frac{5}{2}$

Solution

Consider the system of equations $x^2+x^2y^2+x^2y^4=525$ (1)

and $x+xy+xy^2=35$ (2)

The expression on the left side of equation (1) can be rewritten as,

$$\begin{aligned} x^2+x^2y^2+x^2y^4 &= \left(x+xy^2\right)^2-x^2y^2 \\ &= \left(x+xy^2-xy\right)\left(x+xy^2+xy\right) \end{aligned}$$

Thus, $\left(x+xy^2-xy\right)\left(x+xy^2+xy\right)=525$

Substituting from (2) gives, $\left(x+xy^2-xy\right)(35)=525$

or, $x+xy^2-xy=15$ (3)

Now subtracting (3) from (2), $2xy=20$, $x=\frac{10}{y}$

Substituting for x in (3) gives,

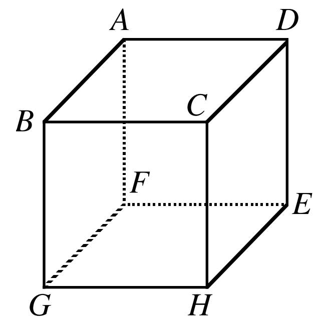
$$\frac{10}{y}+10y-10=15$$

$$10y^2 - 25y + 10 = 0$$
$$2y^2 - 5y + 2 = 0$$
$$(2y-1)(y-2) = 0$$
$$y = \frac{1}{2} \quad \text{or} \quad y = 2$$

The sum of the real y values satisfying the system is $\frac{5}{2}$.

ANSWER: (E)

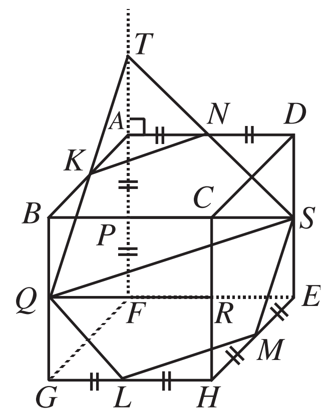
25. The given cube is cut into four pieces by two planes. The first plane is parallel to face $ABCD$ and passes through the midpoint of edge BG. The second plane passes through the midpoints of edges AB, AD, HE, and GH. Determine the ratio of the volumes of the smallest and largest of the four pieces.



(**A**) 3:8 (**B**) 7:24 (**C**) 7:25
(**D**) 7:17 (**E**) 5:11

Solution

For convenience, let each edge of the cube have length 2. The plane *PQRS* through the mid-point of *BG* and parallel to face *ABCD* bisects the volume of the cube. The plane through *K, L, M,* and *N* also bisects the volume of the cube and contains the line segment *QS*. Hence the two planes divide the cube into four pieces, two equal 'smaller' pieces and two equal 'larger' pieces. Extend *QK* and *SN* to meet at *T*.



We note that ΔTAN is similar to ΔTPS. Since $\frac{TA}{TP} = \frac{AN}{PS}$ and $AN = 1$, $PS = 2$ and $PA = 1$ it is easy to calculate to find $TA = 1$.

The volume of the upper 'smallest' piece is equal to the volume of the tetradedron *TQSP* - volume of the tetradedron *TKNA*. This volume is, $\frac{1}{3}\left[\left(\frac{1}{2}(2)(2)(2)\right)\right] - \frac{1}{3}\left[\left(\frac{1}{2}\right)(1)(1)(1)\right]$

$$= \frac{4}{3} - \frac{1}{6} = \frac{7}{6}.$$

The volume of a 'largest' piece is, $2 \times 2 \times 1 - \frac{7}{6} = \frac{17}{6}$.

The ratio of the volume of the 'smallest' piece to that of the 'largest' piece is 7:17.

ANSWER: (D)