Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

1999 Solutions

Euclid Contest

(Grade12)

for the

NATIONAL BANK OF CANADA

Awards

1. (a) If $x^{-1} = 3^{-1} + 4^{-1}$, what is the value of x?

Solution

$$\frac{1}{x} = \frac{1}{3} + \frac{1}{4}$$

$$\frac{1}{x} = \frac{7}{12}$$

$$\therefore x = \frac{12}{7}$$

(b) If the point $P(-3, 2)$ is on the line $3x + 7ky = 5$, what is the value of k?

Solution

Since P is on the line, its coordinates must satisfy the equation of the line.

Thus, $3(-3) + 7k(2) = 5$

$$14k = 14$$

$$k = 1$$

(c) If $x^2 - x - 2 = 0$, determine all possible values of $1 - \frac{1}{x} - \frac{6}{x^2}$.

Solution 1

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$\therefore x = 2 \text{ or } x = -1$$

Substituting

For $x = 2$, $1 - \frac{1}{2} - \frac{3}{2}$

$= -1$

For $x = -1$, $1 + 1 - 6$

$= -4$

Solution 2

$$1 - \frac{1}{x} - \frac{6}{x^2} = \frac{x^2 - x - 6}{x^2}$$

$$= \frac{\left(x^2 - x - 2\right) - 4}{x^2}$$

$$= \frac{-4}{x^2} \quad (\text{since } x^2 - x - 2 = 0)$$

But $x^2 - x - 2 = (x-2)(x+1) = 0$ $\therefore x = 2$ or $x = -1$.

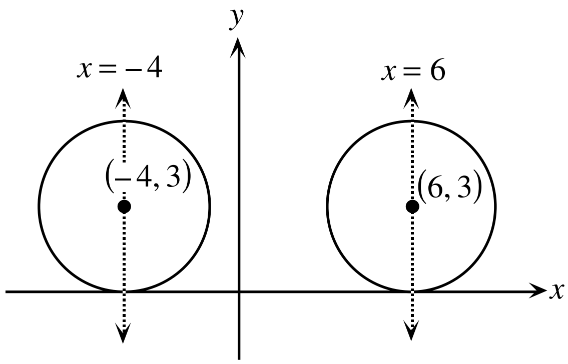
Substituting $x = 2$, <u>or</u> $x = -1$,

$\frac{-4}{4}$ $\qquad$ $= \frac{-4}{1}$

$= -1.$ $\qquad$ $= -4$

2. (a) The circle defined by the equation $(x+4)^2+(y-3)^2=9$ is moved horizontally until its centre is on the line $x=6$. How far does the centre of the circle move?

Solution
The identifying centre is $(-4, 3)$.
Therefore the centre moves 10 units.

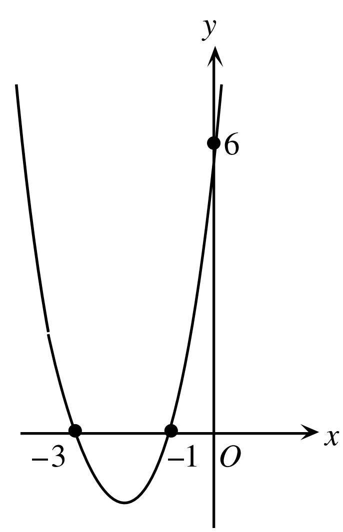


(b) The parabola defined by the equation $y=(x-1)^2-4$ intersects the x-axis at the points P and Q. If (a, b) is the mid-point of the line segment PQ, what is the value of a?

Solution 1
Intercepts of parabola are 3 and –1.
Midpoint is 1, $\therefore a=1$.

Solution 2
Axis of symmetry is $x=1$.
(or vertex is at $(1, -4)$)
By symmetry, $a=1$.

(c) Determine an equation of the quadratic function shown in the diagram.



Solution 1

Let the equation of the parabola be $y = ax^2 + bx + c$.
The parabola passes through the points $(-3, 0)$, $(-1, 0)$ and $(0, 6)$.
Substituting $(0, 6)$ gives, $6 = 0 + 0 + c$, $c = 6$.
Substituting $(-3, 0)$ and $(-1, 0)$ gives,

$$0 = 9a - 3b + 6$$

and $\quad 0 = a - b + 6$.
Solving gives, $a = 2$, $b = 8$.
The equation is $y = 2x^2 + 8x + 6$.

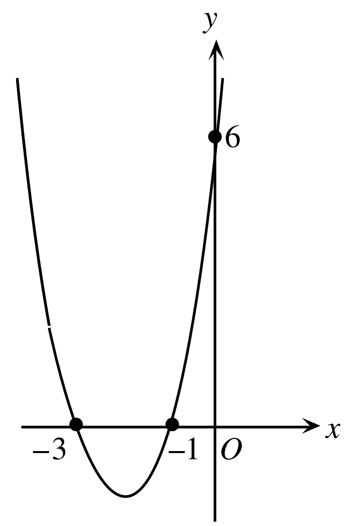
Solution 2

The general form of the quadratic function is, $y = k(x + 3)(x + 1)$.
Since $(0, 6)$ is on the parabola,

$$6 = k(0 + 3)(0 + 1)$$

$$\therefore k = 2.$$

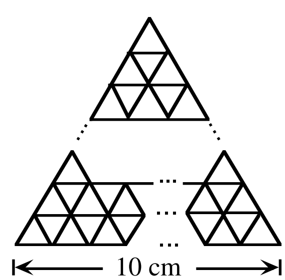
The equation is $y = 2(x + 3)(x + 1)$.



Solution 3

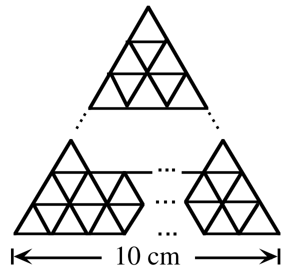
Let the equation of the parabola be $y = a(x + 2)^2 + c$.
Since $(0, 6)$ is on parabola, $6 = 4a + c$,
and $(-1, 0)$ is on parabola, $\quad 0 = a + c$.
Solving, $a = 2$, $c = -2$.
$\therefore$ Equation is $y = 2(x + 2)^2 - 2$.

3. (a) How many equilateral triangles of side 1 cm, placed as shown in the diagram, are needed to completely cover the interior of an equilateral triangle of side 10 cm?



Solution1

If we proceed by pattern recognition, we find after row 1 we have a total of 1 triangle, after two rows we have 2^2 or 4 triangles. After ten rows we have 10^2 or 100 triangles.



Solution 2

This solution is based on the fact that the ratio of areas for similar triangles is the square of the ratio of corresponding sides. Thus the big triangle with side length ten times that of the smaller triangle has 100 times the area.

(b) The populations of Alphaville and Betaville were equal at the end of 1995. The population of Alphaville decreased by 2.9% during 1996, then increased by 8.9% during 1997, and then increased by 6.9% during 1998. The population of Betaville increased by r % in each of the three years. If the populations of the towns are equal at the end of 1998, determine the value of r correct to one decimal place.

Solution

If P is the original population of Alphaville and Betaville,

$$P(.971)(1.089)(1.069) = P\left(1+\frac{r}{100}\right)^3 \qquad (1)$$

$$1.1303 = \left(1+\frac{r}{100}\right)^3 \qquad (2)$$

From here,

Possibility 1

$$1+\frac{r}{100} = (1.1303)^{\frac{1}{3}}$$

$$1+\frac{r}{100} = 1.0416$$

$$r \doteq 4.2\%$$

Or, ***Possibility 2***

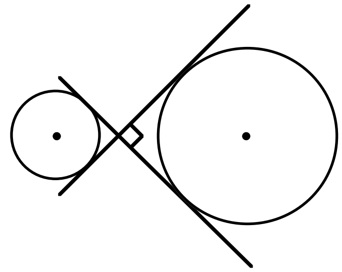
$$3\log\left(1+\frac{r}{100}\right) = \log 1.1303$$

$$\log\left(1+\frac{r}{100}\right) = .01773$$

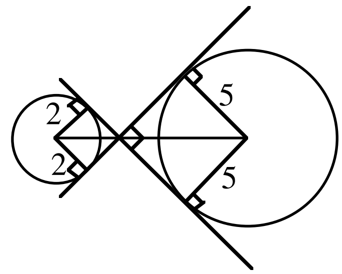
$$1+\frac{r}{100} = 1.0416$$

$$r \doteq 4.2\%$$

4. (a) In the diagram, the tangents to the two circles intersect at $90°$ as shown. If the radius of the smaller circle is 2, and the radius of the larger circle is 5, what is the distance between the centres of the two circles?

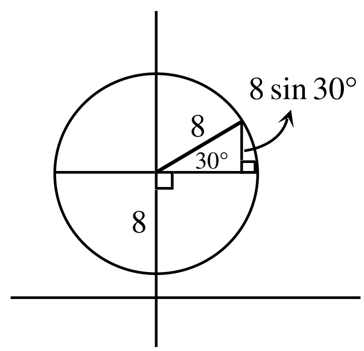


Solution
The distance from the centre of smaller circle to the point of intersection is $2\sqrt{2}$.
The distance from the centre of larger circle to the point of intersection is $5\sqrt{2}$.
Therefore the total distance is $7\sqrt{2}$.

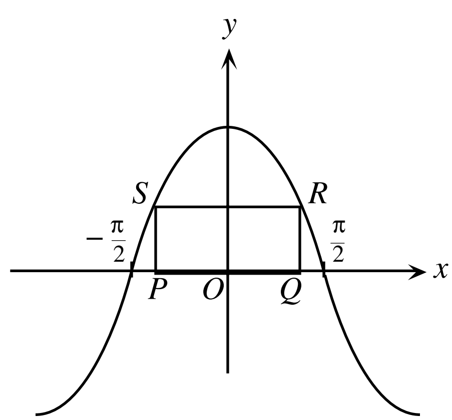


(b) A circular ferris wheel has a radius of 8 m and rotates at a rate of $12°$ per second. At $t = 0$, a seat is at its lowest point which is 2 m above the ground. Determine how high the seat is above the ground at $t = 40$ seconds.

Solution
At $t = 40$, the seat would have rotated $480°$ or $120°$ from its starting position.
We draw the triangle as shown.
The height above the ground is
$2 + 8 + 8 \sin 30°$
$= 14$ m.



5. (a) A rectangle $PQRS$ has side PQ on the x-axis and touches the graph of $y = k \cos x$ at the points S and R as shown. If the length of PQ is $\frac{\pi}{3}$ and the area of the rectangle is $\frac{5\pi}{3}$, what is the value of k?

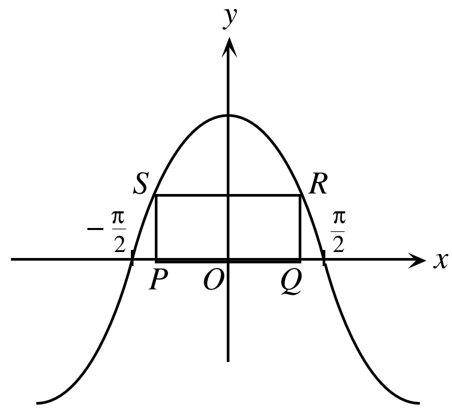


Solution

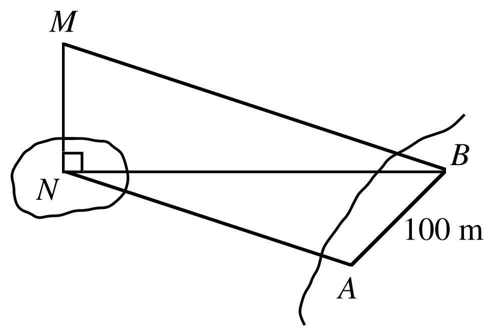
If $PQ = \frac{\pi}{3}$, then by symmetry the coordinates of R are $\left(\frac{\pi}{6}, k\cos\frac{\pi}{6}\right)$.

Area of rectangle $PQRS = \frac{\pi}{3}\left(k\cos\frac{\pi}{6}\right) = \frac{\pi}{3}(k)\left(\frac{\sqrt{3}}{2}\right)$

But $\frac{\sqrt{3}k\,\pi}{6} = \frac{5\pi}{3}$ $\quad \therefore k = \frac{10}{\sqrt{3}}$ or $\frac{10}{3}\sqrt{3}$.



(b) In determining the height, MN, of a tower on an island, two points A and B, 100 m apart, are chosen on the same horizontal plane as N. If $\angle NAB = 108°$, $\angle ABN = 47°$ and $\angle MBN = 32°$, determine the height of the tower to the nearest metre.



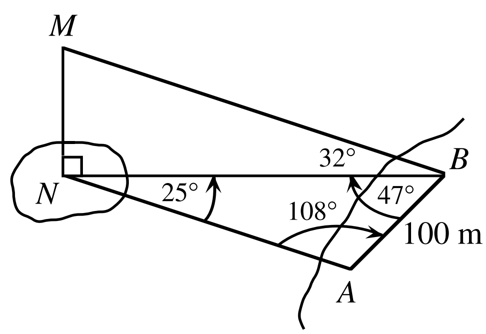
Solution

In ΔBAN, $\angle BNA = 25°$

Using the Sine Law in ΔBAN,

$\frac{NB}{\sin 108°} = \frac{100}{\sin 25°}$

Therefore $NB = \frac{100\sin 108°}{\sin 25°} \approx 225.04$,

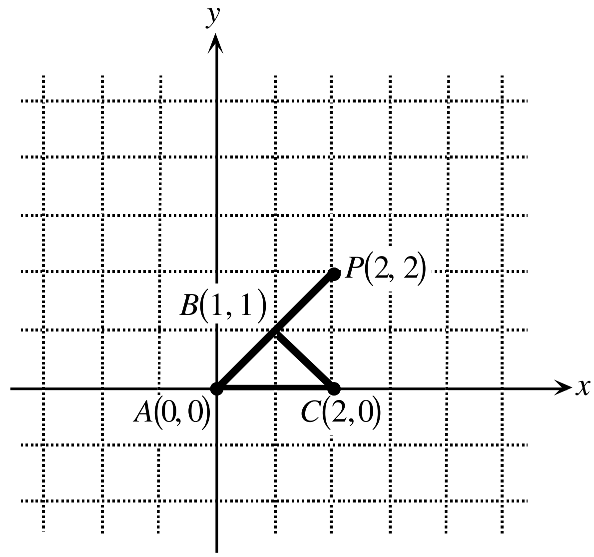


Now in ΔMNB, $\frac{MN}{NB} = \tan 32°$

$$MN = \frac{100\sin 108°}{\sin 25°} \times \tan 32° \doteq 140.6$$

The tower is approximately 141 m high.

6. (a) The points A, P and a third point Q (not shown) are the vertices of a triangle which is similar to triangle ABC. What are the coordinates of all possible positions for Q?

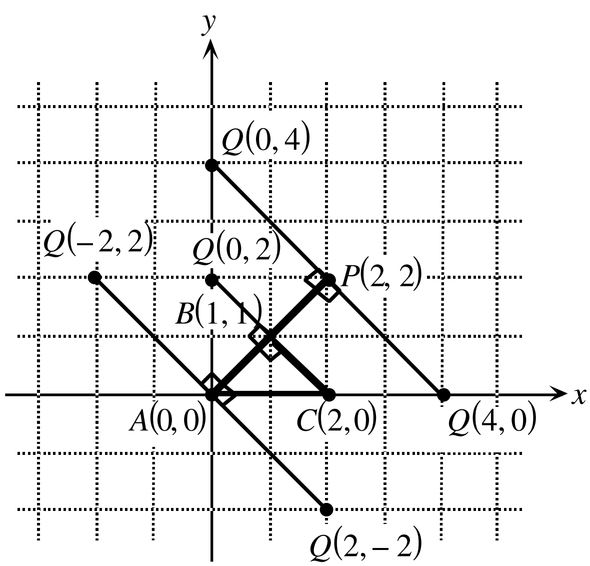


Solution

$Q(4,0)$, $Q(0,4)$

$Q(2,0)$, $Q(0,2)$

$Q(-2,2)$, $Q(2,-2)$



(b) Determine the coordinates of the points of intersection of the graphs of $y=\log_{10}(x-2)$ and $y=1-\log_{10}(x+1)$.

Solution

The intersection takes place where,

$$\log_{10}(x-2)=1-\log_{10}(x+1)$$

$$\log_{10}(x-2)+\log_{10}(x+1)=1$$

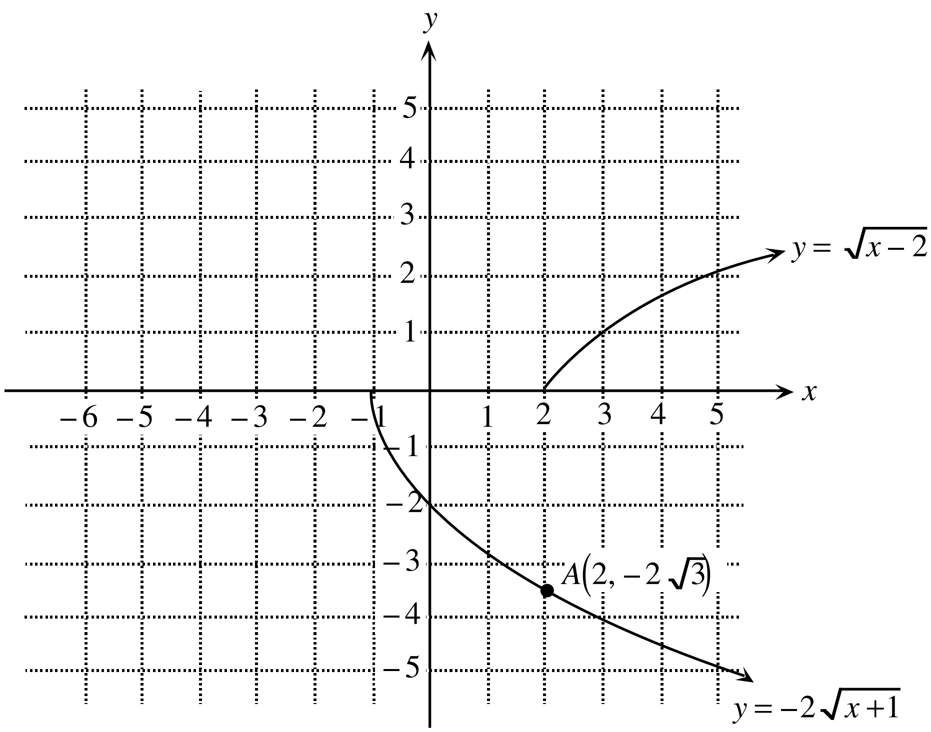
$$\log_{10}\left(x^2-x-2\right)=1$$

$$x^2 - x - 2 = 10$$
$$x^2 - x - 12 = 0$$
$$(x-4)(x+3) = 0$$
$$x = 4 \ \ or \ \ -3$$

For $x = -3$, y is not defined.
For $x = 4$, $y = \log_{10} 2 \doteq 0.3$.
The graphs therefore intersect at $\left(4, \log_{10} 2\right)$.

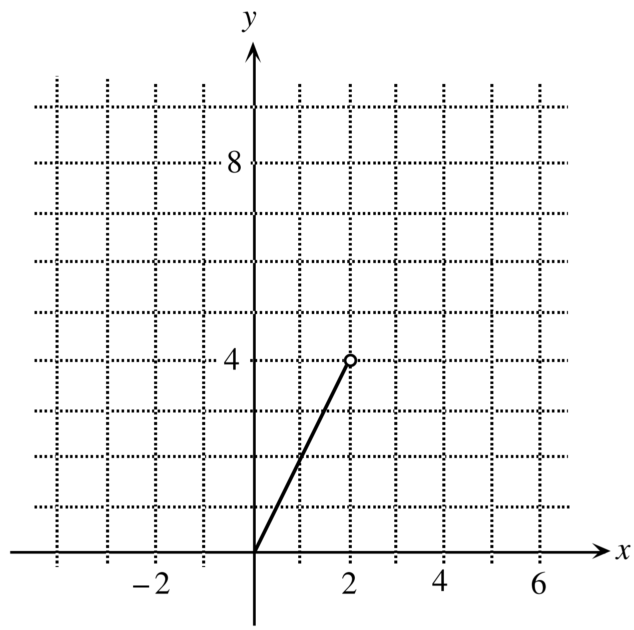
7. (a) On the grid provided in the answer booklet, draw the graphs of the functions $y = -2\sqrt{x+1}$ and $y = \sqrt{x-2}$. For what value(s) of k will the graphs of the functions $y = -2\sqrt{x+1}$ and $y = \sqrt{x-2} + k$ intersect? (Assume x and k are real numbers.)

Solution

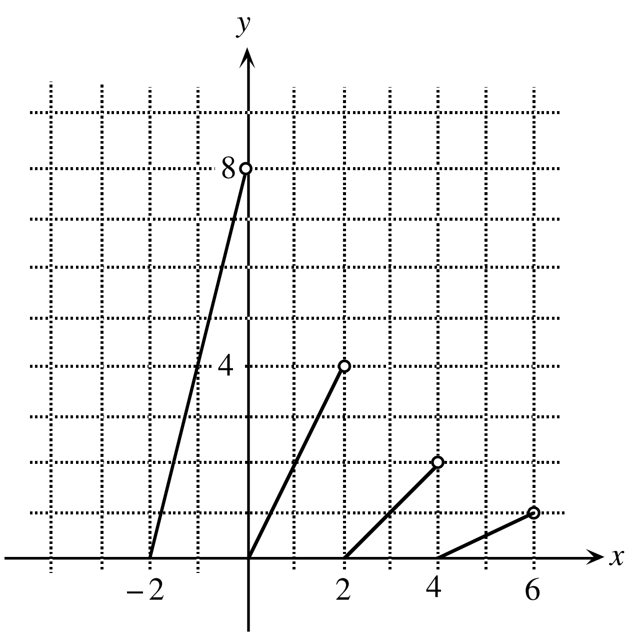


The graph of $y = \sqrt{x-2} + k$ is identical in size and shape compared to that of $y = \sqrt{x-2}$. The parameter, k, just means that the graph can take any position on the line $x = 2$ for all real values of k. If we allow $y = \sqrt{x-2} + k$ to move and it slides down to point A, this implies that for $k = -2\sqrt{3}$ the graphs will intersect. The graphs will also intersect for $k < -2\sqrt{3}$ which is easily seen graphically. Thus the required values for k are $k \le -2\sqrt{3}$, .

(b) Part of the graph for $y = f(x)$ is shown, $0 \le x < 2$.
If $f(x+2) = \frac{1}{2} f(x)$ for all real values of x, draw the graph for the intervals, $-2 \le x < 0$ and $2 \le x < 6$.



Solution



In (b) students did not know how to use the functional equation, $f(x+2) = \frac{1}{2} f(x)$. We will give an example to indicate how we might use this notation to get the required graph.

Let $x = 1$ which gives us $f(3) = \frac{1}{2} f(1)$ when we substitute into the given equation. Since $f(1) = 2$ (which we read from the given graph) we can say that $f(3) = \frac{1}{2}(2) = 1$. This means that (3, 1) is a point on the new graph. We can proceed from here by considering different values for x.

8. (a) The equation $y = x^2 + 2ax + a$ represents a parabola for all real values of a. Prove that each of these parabolas pass through a common point and determine the coordinates of this point.

Solution 1

Since $y = x^2 + 2ax + a$ for all a, $a \in R$, it must be true for $a = 0$ and $a = 1$.
For $a = 0$, $y = x^2$; for $a = 1$, $y = x^2 + 2x + 1$.

By comparison, (or substitution) $x^2 = x^2 + 2x + 1$

$$\therefore x = \frac{-1}{2}$$

$$\Rightarrow y = \frac{1}{4}$$

We must verify that $x = \frac{-1}{2}$, $y = \frac{1}{4}$ satisfies the original.

Verification: $y = x^2 + 2ax + a = \left(\frac{-1}{2}\right)^2 + 2a\left(\frac{-1}{2}\right) + a = \frac{1}{4} - a + a = \frac{1}{4}$

$$\therefore \left(\frac{-1}{2}, \frac{1}{4}\right) \text{ is a point on } y = x^2 + 2ax + a,\ a \in R.$$

Note:

Students can choose values other than $a = 0$, $a = 1$ to achieve the same result.

Solution 2

If $y = x^2 + 2ax + a$ represents a parabola for all real values of a then it is true for all a and b where $a \neq b$.
So, $y = x^2 + 2ax + a$ and $y = x^2 + 2bx + b$ (by substitution of a and b into $y = x^2 + 2ax + a$)
Since we are looking for common point, $x^2 + 2ax + a = x^2 + 2bx + b$

$2ax - 2bx + a - b = 0$
$a(2x + 1) - b(2x + 1) = 0$
$(a - b)(2x + 1) = 0$

Since $a \neq b$, $2x + 1 = 0 \Rightarrow x = \frac{-1}{2}$ and $y = \frac{1}{4}$.

Solution 3

(1) The parabola can be written as, $y = x^2 + a(2x + 1)$.

(2) If $2x + 1 = 0$, then $x = \frac{-1}{2}$ and $y = \frac{1}{4}$ by substitution

Line (2) is true for all values of a and hence $\left(\frac{-1}{2}, \frac{1}{4}\right)$ is a point that is always on the given parabola.

Solution 4

Let the common point be (p, q) for all a.

$\therefore \quad p = q^2 + 2ap + a$

For $a = 0$, $q = p^2$

For $a = 1$, $q = p^2 + 2p + 1$

$\therefore \quad 2p + 1 = 0$, $p = \frac{-1}{2}$ and $q = \frac{1}{4}$

Hence the point is $\left(\frac{-1}{2}, \frac{1}{4}\right)$.

Verification as in Solution 1.

(b) The vertices of the parabolas in part (a) lie on a curve. Prove that this curve is itself a parabola whose vertex is the common point found in part (a).

Solution

Calculating the coordinates of the vertex of $y = x^2 + 2ax + a$,

$$y = x^2 + 2ax + a^2 - a^2 + a$$

$$y = (x + a)^2 - a^2 + a$$

$\therefore$ Vertex is $\left(-a, -a^2 + a\right)$.

We can determine the required by letting $x = -a$ and $y = -a^2 + a$.

Substitute $a = -x$ into $y = -a^2 + a$ to obtain $y = -x^2 - x$.

Completing the square of $y = -x^2 - x$ gives $y = -\left(x + \frac{1}{2}\right)^2 + \frac{1}{4}$.

$\therefore \left(\frac{-1}{2}, \frac{1}{4}\right)$ is the vertex of $y = -x^2 - x$.

9. A 'millennium' series is any series of consecutive integers with a sum of 2000. Let m represent the first term of a 'millennium' series.

(a) Determine the minimum value of *m*.

(b) Determine the smallest possible positive value of *m*.

Solution 1 - Parts (a) and (b)

Series is, $m + (m+1) + (m+2) + \ldots + \left(m + (k-1)\right) = 2000$

Therefore, $\frac{(m+k-1)(m+k)}{2} - \frac{(m-1)m}{2} = 2000$

$$k(2m + k - 1) = 4000$$

Parity Argument

If *k* is odd then $2m + k - 1$ is even and vice-versa.

(Note: This is true because if *k* is odd then $k - 1$ is even as is $2m$ so $2m + k - 1$, is itself even. A similar argument can be made for *k* an even integer to show that $2m + k - 1$ is odd.)

One of the factors of 4000 must be 1, 5, 25 or 125 which gives the eight cases listed below.

Note:

If we had used $m+(m+1)+...+(m+k)=2000$, we would have arrived at $(k+1)(2m+k)=4000$ and then the parity argument is virtually identical to that presented just above.

Listing of Possibilities

k	$2m+k-1$	m
1	4000	2000
5	800	398
25	160	68
125	32	–46
4000	1	–1999
800	5	–397
160	25	–67
32	125	47

(a) minimum value of m is -1999

(b) smallest possible positive value of m is 47

Solution 2 - Parts (a) and (b)

Note that this argument is very similar to the previous but initially it looks different.

With n integers in the series: $m+m+1+...+[m+(n-1)]=2000$

$$\frac{1}{2}n(n+2m-1)=2000$$

$$n^2+(2m-1)n-4000=0$$

Since n is a positive integer this expression factors.

Since the sum of the roots is $-(2m-1)$, an odd integer, the roots must be one odd and one even.

The product of the roots is 4000.

The odd divisors of 4000 are 1, 5, 25 *or* 125.

Factorization	$(2m-1)$	m
$(n-1)(n+4000)$	3999	2000
$(n-5)(n+800)$	795	398
$(n-25)(n+160)$	135	68
$(n-125)(n+32)$	-93	-46
$(n+1)(n-4000)$	-3999	–1999
$(n+5)(n-800)$	-795	-397
$(n+25)(n-160)$	-135	-67
$(n+125)(n-32)$	93	47

(a) –1999

(b) 47

Possible Solution - Part (a)

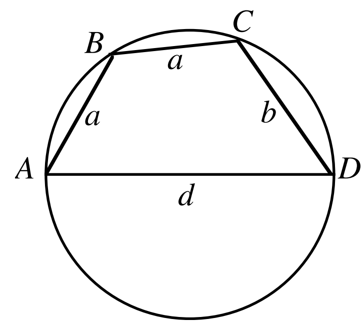
If we start with a negative number m and add consecutive integers the sum will remain negative until we have added the integers from m to $|m|$ at which time the sum will be 0. To reach a positive sum we add one more term, giving us a sum of $|m|+1$. Thus if we add the numbers -1999, ..., 1999, 2000 we get a sum of 2000. However, if we start with an integer m less than -1999 and add until we get to $|m|+1$ the sum will be greater than 2000, and will get even larger if we add further integers. Thus the minimum value for m is -1999.

This argument recognizes that -1999 is an answer but also justifies that $m=-1999$ is the minimum answer.

10. $ABCD$ is a cyclic quadrilateral, as shown, with side $AD=d$, where d is the diameter of the circle. $AB=a$, $BC=a$ and $CD=b$. If a, b and d are integers $a \neq b$,

(a) prove that d cannot be a prime number.

(b) determine the *minimum* value of d.



Solution

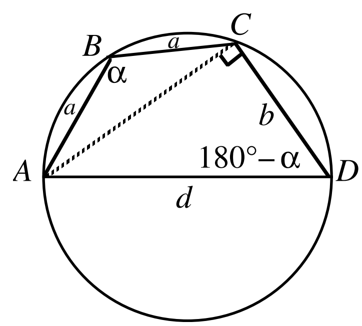
(a) Join A to C and since $\angle ACD$ is in a semicircle, $\angle ACD = 90°$.

Let $\angle ABC = \alpha$, $\therefore \angle CDA = 180° - \alpha$ (cyclic quad.)

From ΔABC, $AC^2 = a^2 + a^2 - 2a^2 \cos\alpha$.

From ΔACD, $AC^2 = d^2 - b^2$ and $\cos(180° - \alpha) = \dfrac{b}{d}$

or $\cos\alpha = \dfrac{-b}{d}$.



By substitution, $d^2 - b^2 = 2a^2 - 2a^2\left(\dfrac{-b}{d}\right)$

$$d^3 - db^2 = 2a^2d + 2a^2b$$

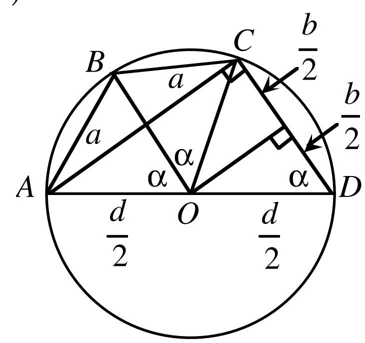
$$d\left(d^2 - b^2\right) = 2a^2(d+b)$$

$$2a^2 = d(d-b),\ d \neq b$$

Note that this relationship could also be reached in the following way:

A Possible Second Method for deriving, $2a^2 = d(d-b)$

Using ΔOBC for example,



From ΔOBC, $a^2 = \frac{d^2}{4} + \frac{d^2}{4} - 2\left(\frac{d}{2}\right)\left(\frac{d}{2}\right)\cos\alpha$

$$\therefore a^2 = \frac{d^2}{2}(1-\cos\alpha)$$

But $\cos\alpha = \frac{\frac{b}{2}}{\frac{d}{2}} = \frac{b}{d}$.

By substitution, $2a^2 = d(d-b)$ as before.

From here we can use the above relationships to arrive at a contradiction.
From $2a^2 = d(d-b)$, if we start by assuming d is a prime then $d = 2$ or $d \geq 3$.

Case 1 $d = 2$

If we make the substitution $d = 2$ in $2a^2 = d(d-b)$ then we have,

$$2a^2 = 2(2-b)$$
$$a^2 = 2-b$$
$$b + a^2 = 2.$$

Since a and b are integers then this implies $a = b = 1$ which is not possible since we are told that a and b must be different.

Case 2 $d \geq 3$, d a prime

Students should start by looking at the relationship, $2a^2 = d(d-b)$. If $d \geq 3$ and we look at the left and right side of this relationship then d must divide into the left hand side. Since $d > 2$, it is not possible that $d|2$. So clearly, then, d must divide a^2. However, we made the assumption that d is prime and greater than 2 so d not only divides a^2 but d also divides a. This is not possible, however, since d is the diameter of the circle and is larger than a. Our original assumption that d was prime must be incorrect and so d must be a composite number.

Note:
Students should observe that there was nothing in our proof that prevents d from being a composite number. In part (b) an example is given to show this possibility.

Solution

(b) Note that d is not prime so $d \neq 2, 3, 5, 7$, etc.

Try $d = 4$, $2a^2 = 4(4-b)$

$$a^2 = 2(4-b).$$

If $b = 1$ or 3 then $a^2 = 6$ or 2 so a is not an integer.

If $b = 2$ then $a = 2$ but $a \neq b$ so this is not possible.

Try $d = 6$, $2a^2 = 6(6 - b)$, $a^2 = 3(6 - b)$.
If $b = 1$, 2, 4 or 5 then a is not an integer.
If $b = 3$ then $a = 3$ but $a \neq b$ as before.

Try $d = 8$, $2a^2 = 8(8 - b)$, $a^2 = 4(8 - b)$.
$\therefore b = 7$ gives $a = 2$, an acceptable solution.
So the minimum value of d is 8.