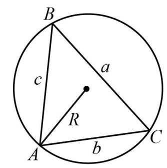
# Trigonometry

## Toolkit

| *Name* | *Formula* |
|---|---|
| Sine Law | $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$, where $R$ is the radius of the circumcircle.   |
| Cosine Law | $a^2 = b^2 + c^2 - 2bc\cos A$<br>$b^2 = a^2 + c^2 - 2ac\cos B$<br>$c^2 = b^2 + a^2 - 2ab\cos C$ |
| Area relations | The area of triangle $ABC$ is $\frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$. |
| General Identities | $\cot\theta = \dfrac{1}{\tan\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\csc\theta = \dfrac{1}{\sin\theta}$<br>$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$<br>$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$ |
| Pythagorean Identities | $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, $\cot^2\theta + 1 = \csc^2\theta$ |
| Sum Formulas | $\sin(A+B) = \sin A\cos B + \cos A\sin B$<br>$\cos(A+B) = \cos A\cos B - \sin A\sin B$ |
| Double Angle Formulas | $\sin(2\theta) = 2\sin\theta\cos\theta$<br>$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ |

## Sample Problems

1. Determine the values of $x$ such that $2\sin^3 x - 5\sin^2 x + 2\sin x = 0$ given that $0 \le x \le 2\pi$.

   **Solution**

   We factor the given equation to obtain

$$\sin x(2\sin^2 x - 5\sin x + 2) = 0$$
$$\sin x(2\sin x - 1)(\sin x - 2) = 0$$

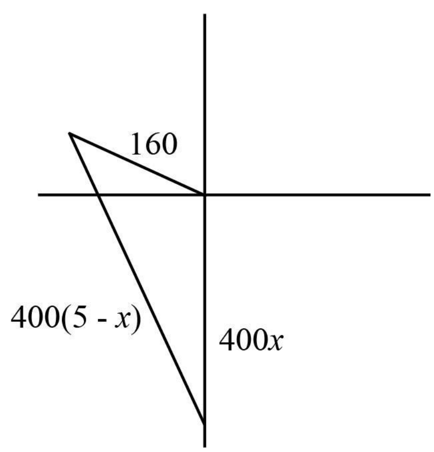
   So $\sin x = 0$, $\frac{1}{2}$ or 2. But $|\sin x| \le 1$. So $\sin x \ne 2$. Therefore, in the interval $0 \le x \le 2\pi$, we have $x = 0$, $\pi$, $2\pi$, $\frac{\pi}{6}$ or $\frac{5\pi}{6}$.

2. An airplane leaves an aircraft carrier and flies due south at 400 km/hr. The carrier proceeds at a heading of 60° west of north at 32 km/hr. If the plane has 5 hours of fuel, what is the maximum distance south the plane can travel so that the fuel remaining will allow a safe return to the carrier at 400 km/hr?

   **Solution**

   The first step in solving this problem is to draw a diagram (as shown). If we let $x$ be the number of hours that the plane flies south, then the distance that the plane flies south is $400x$. The plane then flies a distance $400(5 - x)$ in the remaining time, while the total distance the carrier travels is $5(32) = 160$. Using these distances, the cosine law gives us

$$(400(5 - x))^2 = 160^2 + (400x)^2 - 2 \cdot 160 \cdot 400x \cdot \cos 120°.$$

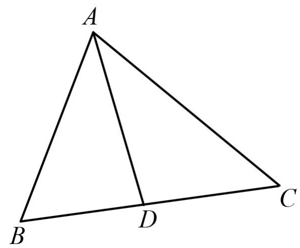



   Simplifying we obtain

$$4000000 - 1600000x + 160000x^2 = 25600 + 160000x^2 + 64000x$$

   which we can solve to get $x = \frac{621}{260}$. Thus, the maximum distance the plane should travel south is $400\left(\frac{621}{260}\right) = \frac{12420}{13}$ km, which is approximately 955.4 km.

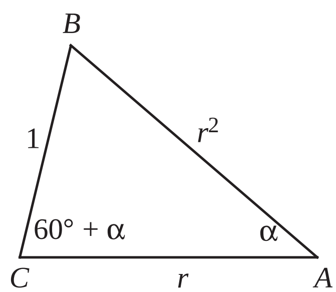
3. In triangle $ABC$, the point $D$ is on $BC$ such that $AD$ bisects $\angle A$. Show that $\dfrac{AB}{BD} = \dfrac{AC}{CD}$.




**Solution**

We call $\angle ADC = \theta$ and $\angle BAC = \alpha$. We use the sine law in triangles $ADC$ and $ADB$ to obtain $\dfrac{\sin\frac{\alpha}{2}}{\sin\theta} = \dfrac{CD}{AC}$ and $\dfrac{\sin\frac{\alpha}{2}}{\sin(180^\circ - \theta)} = \dfrac{BD}{AB}$. But $\sin\theta = \sin(180^\circ - \theta)$ and so $\dfrac{AB}{BD} = \dfrac{AC}{CD}$. This result is known as the angle bisector theorem.

4. For the given triangle $ABC$, $\angle C = \angle A + 60^\circ$. If $BC = 1$, $AC = r$ and $AB = r^2$, where $r > 1$, prove that $r < \sqrt{2}$.




**Solution**

We represent the angles of the triangle as: $\angle A = \alpha$, $\angle C = \alpha + 60^\circ$, and $\angle B = 120^\circ - 2\alpha$. So the sine law states

$$\begin{aligned}\frac{r^2}{1} &= \frac{\sin(\alpha + 60^\circ)}{\sin\alpha} \\ &= \frac{\sin\alpha\cos 60^\circ + \cos\alpha\sin 60^\circ}{\sin\alpha} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\cot\alpha\end{aligned}$$

Since all three angles in the triangle are positive, we can see that $0 < \alpha < 60^\circ$ . In this range, the tangent function is increasing, and its reciprocal, the cotangent function, is decreasing.

The cosine law gives

$$r^2 = 1 + r^4 - 2r^2\cos(120^\circ - 2\alpha).$$

But $r > 1$ and so

$$(r^2 - 1)^2 > 0 \text{ or } r^4 + 1 > 2r^2.$$

Substituting the second inequality into the equation gives $r^2 > 2r^2 - 2r^2\cos(120° - 2\alpha)$ which implies $\cos(120° - 2\alpha) > \frac{1}{2}$. Thus, $\alpha > 30°$ and

$$r^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}\cot\alpha < \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{1} = 2$$

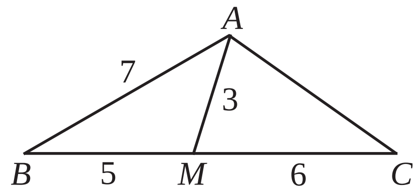
Thus, $r^2 < 2$ and so $r < \sqrt{2}$.

## Problem Set

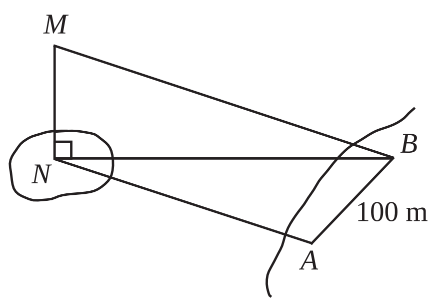
1. (a) If $2\sin(2\theta) + 1 = 0$, find the smallest positive value of $\theta$ (in degrees).

   (b) For $-\pi \leq \theta \leq \pi$, find all solutions to the equation $2(\sin^2\theta - \cos^2\theta) = 8\sin\theta - 5$.

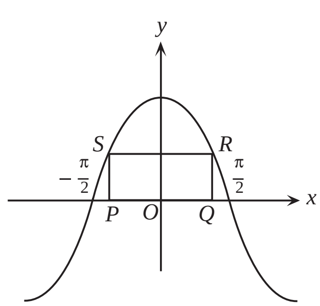
2. In $\triangle ABC$, $M$ is a point on $BC$ such that $BM = 5$ and $MC = 6$. If $AM = 3$ and $AB = 7$, determine the exact value of $AC$.



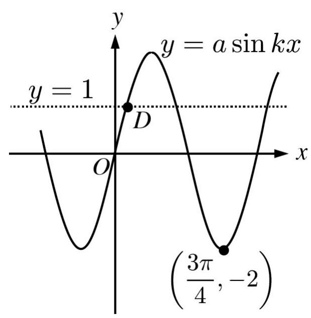

3. In determining the height, $MN$, of a tower on an island, two points $A$ and $B$, 100 m apart, are chosen on the same horizontal plane as $N$. If $\angle NAB = 108°$, $\angle ABN = 47°$ and $\angle MBN = 32°$, determine the height of the tower to the nearest metre.



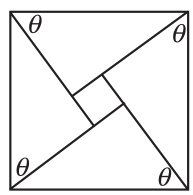

4. A rectangle $PQRS$ has side $PQ$ on the $x$-axis and touches the graph of $y = k\cos x$ at the points $S$ and $R$ as shown. If the length of $PQ$ is $\frac{\pi}{3}$ and the area of the rectangle is $\frac{5\pi}{3}$, what is the value of $k$?



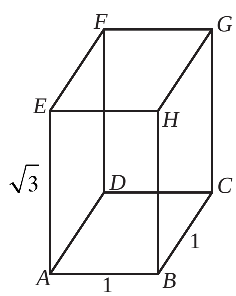
5. The graph of the equation $y = a \sin kx$ is shown in the diagram, and the point $\left(\frac{3\pi}{4}, -2\right)$ is the minimum point indicated. The line $y = 1$ intersects the graph at point $D$. What are the coordinates of $D$?



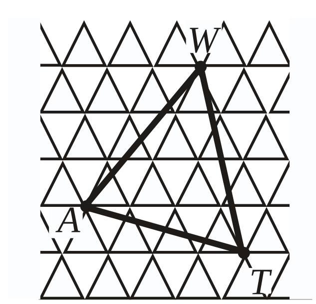

6. A square with an area of 9 cm$^2$ is surrounded by four congruent triangles, forming a larger square with an area of 89 cm$^2$. If each of the triangles has an angle $\theta$ as shown, find the value of $\tan\theta$.




7. A rectangular box has a square base of length 1 cm, and height $\sqrt{3}$ cm as shown in the diagram. What is the cosine of angle $FAC$?




8. In the grid, each small equilateral triangle has side length 1. If the vertices of $\triangle WAT$ are themselves vertices of small equilateral triangles, what is the area of $\triangle WAT$?



9. In $\triangle ABC$, $AB = 8$, and $\angle CAB = 60°$. Sides $BC$ and $AC$ have integer lengths $a$ and $b$, respectively. Find all possible values of $a$ and $b$.