Properties of Numbers

Toolkit

Divisibility Rules

- An integer is divisible by 2 if and only if its final digit is divisible by 2.
- An integer is divisible by 3 if and only if the sum of its digits is divisible by 3.
- An integer is divisible by 4 if and only if the number formed by the last two digits is divisible by 4.
- An integer is divisible by 5 if and only if its final digit is 0 or 5.
- An integer is divisible by 9 if and only if the sum of its digits is divisible by 9.
- An integer is divisible by 10 if and only if its final digit is 0.
- An integer is divisible by 11 if and only if the alternating sum of its digits is divisible by 11.

General Properties of Integers

- A k-digit number $n = d_{k-1}d_{k-2}\cdots d_2d_1d_0$, where d_i is a digit from 0 to 9, can be expressed in the form
$$d_0 + d_1 \times 10 + d_2 \times 10^2 + \cdots + d_{k-2} \times 10^{k-2} + d_{k-1} \times 10^{k-1}$$
We call this expression the *decimal expansion* of n. Each integer n has a unique decimal expansion.
- If integer $p \geq 2$ is prime and $p = ab$, where a and b are positive integers, then either $a = 1$ or $b = 1$.
- An integer n is a perfect square if and only if $n = k^2$ for some non-negative integer k.
- Every integer $n \geq 2$ can be written as a product of prime factors uniquely, apart from the order of factors.
- If $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ is the unique representation of a positive integer $n \geq 2$ as a product of distinct primes $p_1, p_2, \ldots, p_k$, then n has $(a_1 + 1)(a_2 + 1)\cdots(a_k + 1)$ positive divisors.
- Every integer n is either even or odd. If n is even, then $n = 2k$ for some integer k. If n is odd, then $n = 2k + 1$ for some integer k.

  In a similar manner, we can write any integer in one of the following forms: $3k, 3k+1$ or $3k+2$ for some integer k.

  More generally, for all integers a and positive integers b, there exist unique integers q and r such that
$$a = qb + r, \quad 0 \leq r < b$$
- For all integers a and b, and prime numbers p, if p is a divisor of ab, then p is a divisor of a or p is a divisor of b.

Sample Problems

1. Find the smallest positive integer k such that $504k$ is a perfect square.

   Solution

   The prime factorization of 504 is $504 = 2^3 \times 3^2 \times 7$. The prime factorization of a perfect square must include each prime factor an even number of times. Therefore, if $504k$ is a perfect square, then k must be of the form $2 \times 7 \times m$ where m is a perfect square. The smallest positive value of m which is a perfect square is $m = 1$ and so $k = 14$.

2. The number 27572 is a *palindrome* because it reads the same backwards as forwards. What is the largest five-digit palindrome divisible by 6?

   Solution

   An integer is divisible by 6 if it is divisible by both 2 and 3. Since the palindrome is divisible by 2, the last digit must be even. Therefore, the first digit will be even. The largest possible first digit is 8. The largest possible second digit is 9 and so the fourth digit will be 9. To determine the middle digit we use the fact that if an integer is divisible by 3, then the sum of its digits must be divisible by 3.

   Let a represent the middle digit. Since a is a digit, we have that $a \leq 9$. Therefore, the sum of the digits is $8 + 9 + a + 9 + 8 = 34 + a$. The largest possible value of $a \leq 9$ for which $34 + a$ is divisible by 3 is 8. Therefore, the largest five-digit palindrome which is divisible by 6 is 89898.

3. When a positive two-digit number m is multiplied by a positive three-digit number n the result is 21210. Find all possible pairs (m, n).

   Solution

   The prime factorization of 21210 is $21210 = 2 \times 3 \times 5 \times 7 \times 101$.

   Since m is a two-digit number it cannot have 101 as a divisor. Therefore, n is a multiple of 101.

   The prime factorization of n can include at most one 2, at most one 3, at most one 5 and at most one 7.

   Since n is a three-digit number, the possible values for n are 101, 202, 303, 505, 606 and 707. The corresponding values for m are 210, 105, 70, 42, 35 and 30, respectively.

   Since m is a two-digit number, 210 and 105 are not possible. Therefore, the possible pairs (m, n) are $(70, 303), (42, 505), (35, 606)$ and $(30, 707)$.

4. A number has exactly eight positive divisors, including one and the number itself. If two of the divisors are 35 and 77, what is the sum of all eight positive divisors?

   Solution

   Let n be the number. Since n is divisible by $35 = 5 \times 7$ and $77 = 7 \times 11$, it must be of the form $k \times 5^a \times 7^b \times 11^c$ where k, a, b, c are positive integers.

   Let m be the number of positive divisors of k where $m \geq 1$. So the number of positive divisors of n is $m(a+1)(b+1)(c+1) = 8$. Since $a, b, c, \geq 1$, it must be the case that $m = a = b = c = 1$ and so $n = 5 \times 7 \times 11 = 385$.

   The eight positive divisors are $1, 5, 7, 11, 35, 55, 77,$ and 385. Their sum is

   $$1 + 5 + 7 + 11 + 35 + 55 + 77 + 385 = 576$$

5. If m and n are integers, prove that $mn(m^4 - n^4)$ is always divisible by 30.

Proof

Let $T = mn(m^4 - n^4) = mn(m^2 - n^2)(m^2 + n^2) = mn(m - n)(m + n)(m^2 + n^2)$.

To show that T is divisible by 30 we must show that it is divisible by 2, 3 and 5.

If at least one of m or n is even, then T is divisible by 2. If m and n are both odd, then $m+n$ is even, and so T is divisible by 2.

If at least one of m or n is divisible by 3, then T is divisible by 3. If m and n are not divisible by 3, then $m = 3k + 1$ or $m = 3k + 2$ for some integer k. However, if $m = 3k + 2$, then $m = 3k + 3 - 1 = 3(k + 1) - 1$. Since k is an integer, it follows that $k + 1$ is an integer and thus, $m = 3j - 1$ for some integer j. The variable name is irrelevant, so we can simply say that $m = 3k + 1$ or $m = 3k - 1$ (or more concisely that $m = 3k \pm 1$), for some integer k.

Similarly, we can say that $n = 3j \pm 1$ for some integer j.

If $m = 3k + 1$ and $n = 3j + 1$, then $m - n = 3k + 1 - 3j - 1 = 3k - 3j = 3(k - j)$. Since k and j are integers, it follows that $m - n$ will be divisible by 3.

In a similar manner, we can show that if $m = 3k - 1$ and $n = 3j - 1$, then $m - n$ will be divisible by 3.

If $m = 3k + 1$ and $n = 3j - 1$, then $m + n = 3k + 1 + 3j - 1 = 3k + 3j = 3(k + j)$. Since k and j are integers, it follows that $m + n$ will be divisible by 3.

In a similar manner, we can show that if $m = 3k - 1$ and $n = 3j + 1$, then $m + n$ will be divisible by 3.

Therefore, in every case, either $m - n$ or $m + n$ is divisible by 3 and thus, T is divisible by 3.

If at least one of m or n is divisible by 5, then T is divisible by 5. If m and n are not divisible by 5, then $m = 5k + 1$ or $m = 5k + 2$ or $m = 5k + 3$ or $m = 5k + 4$, for some integer k. However, if $m = 5k + 3$, then $m = 5k + 5 - 2 = 5(k + 1) - 2$. Since k is an integer, we have that $k + 1$ is an integer and thus, $m = 5j - 2$, for some integer j. Also, if $m = 5k + 4$, then $m = 5k + 5 - 1 = 5(k + 1) - 1$. Since k is an integer, we have that $k + 1$ is an integer and thus, $m = 5\ell - 1$ for some integer ℓ. Again, since the variable names are irrelevant, we can simply say that $m = 5k \pm 1$ or $m = 5k \pm 2$, for some integer k.

Similarly, we can say that $n = 5j \pm 1$ or $n = 5j \pm 2$, for some integer j.

If $m = 5k \pm 1$, then

$$m^2 = (5k \pm 1)^2 = 25k^2 \pm 10k + 1 = 5(5k^2 \pm 2k) + 1$$

Since k is an integer, we have that $5k^2 \pm 2k$ is an integer and so m^2 is of the form $5a + 1$, for some integer a.

If $m = 5k \pm 2$, then

$$m^2 = (5k \pm 2)^2 = 25k^2 \pm 20k + 4 = 25k^2 \pm 20k + 5 - 1 = 5(5k^2 \pm 4k + 1) - 1$$

Since k is an integer, we have that $5k^2 \pm 4k + 1$ is an integer and so m^2 is of the form $5a - 1$ for some integer a. So $m^2 = 5a \pm 1$.

Similarly, we can show that $n^2 = 5b \pm 1$.

Therefore, in a similar manner to what we did in the divisible by 3 case, we can show that in all cases, $m^2 + n^2$ or $m^2 - n^2$ is divisible by 5 and thus, T will be divisible by 5.

Thus, T is divisible by 2, 3, and 5 and so T is divisible by 30.

Problem Set

1. What is the largest palindrome less than 200 that is the sum of three consecutive integers?

2. When a decimal point is placed between the digits of the two-digit integer n, the resulting number is equal to the average of the digits of n. What is the value of n?

3. Let n be the integer equal to $10^{20} - 20$. What is the sum of the digits of n?

4. In the Fibonacci sequence, $1, 1, 2, 3, 5, \ldots$, each term after the second is the sum of the previous two terms. How many of the first 100 terms of the Fibonacci sequence are odd?

5. Determine the number of positive divisors of 900, including 1 and 900, that are perfect squares.

6. Ellie has two lists, each consisting of 6 consecutive positive integers. The smallest integer in the first list is a, the smallest integer in the second list is b, and $a < b$. She makes a third list which consists of the 36 integers formed by multiplying each number from the first list with each number from the second list. (This third list may include some repeated numbers.) If

   - the integer 49 appears in the third list,
   - there is no number in the third list that is a multiple of 64, and
   - there is at least one number in the third list that is larger than 75,

   determine all possible pairs (a, b).

7. Charles was born in a year between 1300 and 1400. Louis was born in a year between 1400 and 1500. Each was born on April 6 in a year that is a perfect square. Each lived for 110 years. In what year while they were both alive were their ages both perfect squares on April 7?

8. What is the smallest positive integer x for which $\dfrac{1}{32} = \dfrac{x}{10^y}$ for some positive integer y?

9. The average of three consecutive multiples of 3 is a.

   The average of four consecutive multiples of 4 is $a + 27$.

   The average of the smallest and largest of these seven integers is 42. Determine the value of a.

10. Determine all pairs (a, b) of positive integers for which $a^3 + 2ab = 2013$.

11. An auditorium has a rectangular array of chairs. There are exactly 14 teachers seated in each row and exactly 10 students seated in each column. If exactly 3 chairs are empty, prove that there are at least 567 chairs in the auditorium.