Functions, Equations and Polynomials

Solutions

1. Subtract the first equation from the second, rearrange the resulting expression and then factor to obtain

\[-8x + y + xy - 8 = 0\]
\[xy - 8x + y - 8 = 0\]
\[x(y - 8) + y - 8 = 0\]
\[(x + 1)(y - 8) = 0\]

Therefore, \(x = -1\) or \(y = 8\). If \(x = -1\), then substituting into the first equation and solving we obtain that \(y = -9\). If \(y = 8\), then substituting into the first equation and solving we obtain \(x = 4 \pm 2\sqrt{2}\). Therefore, the solutions are \((-1, -9)\) and \((4 \pm 2\sqrt{2}, 8)\).

2. Solution 1

We are asked for the \(x\) value of the midpoint of zeros, which is the \(x\) value of the vertex of the parabola. The equation is written in vertex form already and so \(a = 1\).

Solution 2

Find the \(x\)-intercepts:

\[(x - 1)^2 - 4 = 0\]
\[(x - 1)^2 = 4\]
\[x = 1 \pm 2\]

Thus, \(x = 3\) or \(x = -1\). Thus, \(a = \frac{-1 + 3}{2} = 1\).

3. (a) Consider \(a = 0\) and \(a = 1\) and find the intersection point of the resulting equations, \(y = x^2\) and \(y = x^2 + 2x + 1\). Subtracting the equations we obtain \(0 = 2x + 1\). Therefore, \(x = -\frac{1}{2}\) and so the intersection point is \(\left(-\frac{1}{2}, \frac{1}{4}\right)\). Now substitute \(x = -\frac{1}{2}\) into the general equation. Therefore,

\[y = x^2 + 2ax + a\]
\[= \frac{1}{4} + 2a \cdot \left(-\frac{1}{2}\right) + a\]
\[= \frac{1}{4}\]

Since \(\left(-\frac{1}{2}, \frac{1}{4}\right)\) satisfies the general equation, it is a point on all of the parabolas.
(b) Now \( y = x^2 + 2ax + a = (x + a)^2 + a - a^2 \) and so the vertex is at \((-a, a - a^2)\). If we represent the coordinates of the vertex by \((p, q)\) we have \( p = -a \) and \( q = a - a^2 \) or \( q = -p^2 - p \), the required parabola. Completing the square we obtain

\[
q = - \left( p^2 + p + \frac{1}{4} \right) + \frac{1}{4} = - \left( p + \frac{1}{2} \right)^2 + \frac{1}{4}
\]

and so we see that the vertex of this parabola is \((-\frac{1}{2}, \frac{1}{4})\), the common point found in part (a).

4. Factoring both equations we arrive at:

\[
\begin{align*}
\text{(1)} & \quad p(1 + r + r^2) = 26 \\
\text{(2)} & \quad p^2 r(1 + r + r^2) = 156
\end{align*}
\]

From equation (1) we can see neither of the factors of its left-hand side are 0. Dividing (2) by (1) gives \( pr = 6 \). Substituting this relation back into (1) we get

\[
\begin{align*}
6 \frac{1}{r} + 6 + 6r &= 26 \\
6 - 20r + 6r^2 &= 0 \\
3r^2 - 10r + 3 &= 0 \\
(3r - 1)(r - 3) &= 0
\end{align*}
\]

Therefore, \( r = \frac{1}{3} \) or \( r = 3 \). Hence \((p, r) = (2, 3)\) or \((18, \frac{1}{3})\).

5. We assume, on the contrary, that the coefficients are in geometric sequence. Then \( \frac{b}{a} = \frac{c}{b} \) which implies that \( b^2 = ac \). But now the discriminant \( b^2 - 4ac = -3b^2 < 0 \), so that the roots are not real. Thus, we have a contradiction to the condition set out in the statement of the problem and our assumption is false.

6. Let \( r \) and \( s \) be the integer roots. The equation can be written as

\[
a(x - r)(x - s) = a(x^2 - (r + s)x + rs)
\]

\[
= a(x^2 - a(r + s)x + ars)
\]

\[
= ax^2 + bx + c
\]

with \( b = -a(r + s) \) and \( c = ars \). Since \( a, b, c \) are in arithmetic sequence, we have

\[
c - b = b - a
\]

\[
a + c - 2b = 0
\]

\[
a + ars + 2a(r + s) = 0
\]

\[
1 + rs + 2(r + s) = 0 \quad \text{(we can divide by} a \text{ since} a \neq 0) \]

\[
rs + 2r + 2s + 4 = 3
\]

\[
(r + 2)(s + 2) = 3
\]
Ignoring the order of the factors, we can factor 3 as a product of two integers in two ways: $3 = 1(3)$ or $3 = (-1)(-3)$. Therefore, the two possibilities for the roots of quadratic are: (i) −1 and 1 or (ii) −3 and −5.

7. **Solution 1**

Multiplying out and collecting terms results in $x^4 - 6x^3 + 8x^2 + 2x - 1 = 0$. We look for a factoring with integer coefficients, using the fact that the first and last coefficients are 1 and −1, respectively. So

$$x^4 - 6x^3 + 8x^2 + 2x - 1 = (x^2 + ax + 1)(x^2 + bx - 1)$$

where $a$ and $b$ are undetermined coefficients. However, expanding and comparing coefficients gives $a + b = -6$ and $-a + b = 2$ and $ab = 8$. Since all three equations are satisfied by $a = -4$ and $b = -2$, we have factored the original expression as

$$x^4 - 6x^3 + 8x^2 + 2x - 1 = (x^2 - 4x + 1)(x^2 - 2x - 1)$$

Factoring these two quadratics gives the roots $x = 2 \pm \sqrt{3}$ and $x = 1 \pm \sqrt{2}$.

**Solution 2**

We observe that the original equation is of the form $f(f(x)) = x$, where $f(x) = x^2 - 3x + 1$. Now if we can find $x$ such that $f(x) = x$, then $f(f(x)) = x$. So we solve $f(x) = x^2 - 3x + 1 = x$ which gives the first factor $x^2 - 4x + 1$ above. With polynomial division, we can then determine that

$$x^4 - 6x^3 + 8x^2 + 2x - 1 = (x^2 - 4x + 1)(x^2 - 2x - 1)$$

and continue as in Solution 1.

8. The vertex has $x = 2$ and $y = -16$ and so $A = (2, -16)$. When $y = 0$ we get $0 = x^2 - 4x - 12$ which factors to give us intercepts at −2 and 6. The larger value is 6, and so $B = (6, 0)$. Therefore, we want the line through $(2, -16)$ and $(6, 0)$. Finding the slope of the line and using the second point, the equation of the line is

$$y = \left(\frac{0 + 16}{6 - 2}\right)(x - 6)$$

which simplifies to $y = 4x - 24$.

9. **Solution 1**

Multiplying gives

$$x^2 - (b + c)x + bc = a^2 - (b + c)a + bc$$

$$x^2 - (b + c)x + a(-a + b + c) = 0$$

The roots are

$$x = \frac{b + c \pm \sqrt{(b + c)^2 - 4a(-a + b + c)}}{2}$$

$$= \frac{b + c \pm \sqrt{(b + c)^2 + 4a^2 - 4a(b + c)}}{2}$$

$$= \frac{b + c \pm \sqrt{(b + c - 2a)^2}}{2}$$
Thus, \( x = -a + b + c \) or \( x = a \).

Solution 2
Observe that \( x = a \) is one solution. Rearranging as in the first solution we get

\[
x^2 - (b + c)x + a(-a + b + c) = 0
\]

Using the sum (or the product) of the roots, we determine that other root is \( x = -a + b + c \).

10. Since \( x = -2 \) is a solution of \( x^3 - 7x - 6 = 0 \), we know that \( x + 2 \) is a factor of \( x^3 - 7x - 6 \). Factoring (or using long division) we obtain

\[
x^3 - 7x - 6 = (x + 2)(x^2 - 2x - 3)
= (x + 2)(x + 1)(x - 3)
\]

Thus, the roots are \(-2, -1\) and \(3\).

11. Let the roots be \( r \) and \( s \). Using the sum of the roots and the product of the roots we obtain

\[
r + s = \frac{-4(a - 2)}{4}
= 2 - a
\]

and

\[
rs = \frac{-8a^2 + 14a + 31}{4}
= -2a^2 + \frac{7a}{2} + \frac{31}{4}
\]

Then

\[
r^2 + s^2 = (r + s)^2 - 2rs
= (2 - a)^2 - 2\left(-2a^2 + \frac{7}{2}a + \frac{31}{4}\right)
= 4 - 4a + a^2 + 4a^2 - 7a - \frac{31}{2}
= 5a^2 - 11a - \frac{23}{2}.
\]

It appears that the minimum value should be at the vertex of the parabola \( f(a) = 5a^2 - 11a - \frac{23}{2} \), that is, at \( a = \frac{11}{10} \) (found by completing the square). But we have ignored the condition that the roots are real. The discriminant of the original equation is

\[
B^2 - 4AC = [4(a - 2)]^2 - 4(4)(-8a^2 + 14a + 31)
= 16(a^2 - 4a + 4) + 128a^2 - 224a - 496
= 144a^2 - 288a - 432
= 144(a^2 - 2a - 3)
= 144(a - 3)(a + 1).
\]
Thus, we have real roots only when \( a \geq 3 \) or \( a \leq -1 \). Therefore, \( a = \frac{11}{10} \) cannot be our final answer, since the roots are not real for this value. However \( f(a) = 5a^2 - 11a - \frac{23}{2} \) is a parabola opening up and is symmetrical about its axis of symmetry \( a = \frac{11}{10} \). So we move to the nearest value of \( a \) to the axis of symmetry that gives real roots, which is \( a = 3 \).

12. Let \( g(2) = k \). Since \( f \) and \( g \) are inverse functions, we know that \( f(k) = 2 \). We need to solve

\[
\frac{3k - 7}{k + 1} = 2
\]

\[
3k - 7 = 2(k + 1)
\]

\[
k = 9
\]

Thus, \( g(2) = 9 \).

13. Complete the square to obtain

\[
y = -2x^2 - 4ax + k
\]

\[
= -2(x^2 + 2ax + a^2) + k + 2a^2
\]

\[
= -2(x + a)^2 + k + 2a^2
\]

The vertex is at \((-a, k + 2a^2)\) which we know is \((-2, 7)\). Therefore, solving we obtain \( a = 2 \) and \( k = -1 \).

14. Using the sum and the product of the roots we have the four equations:

\[
a + b = -c
\]

\[
ab = d
\]

\[
c + d = -a
\]

\[
cd = b
\]

Therefore,

\[
-(c + d) + cd = -c
\]

\[
-2c + d = 0
\]

\[
d(c - 1) = 0
\]

But none of \( a, b, c \) or \( d \) are zero, so \( c = 1 \). Then we get \( d = b \). Substituting \( d = b \) into \( ab = d \) we get \( a = 1 \). Then \( d = b = -2 \). Thus, \( a + b + c + d = -2 \).

15. The most common way to do this problem uses calculus. However, we make the substitution \( z = x - 4 \). To get \( y \) in terms of \( z \), try

\[
y = x^2 - 2x - 3
\]

\[
= (x - 4)^2 + 6x - 19
\]

\[
= (x - 4)^2 + 6(x - 4) + 5
\]

\[
= z^2 + 6z + 5
\]
Therefore, the value we want to minimize is \( \frac{y - 4}{(x - 4)^2} = \frac{z^2 + 6z + 1}{z^2} = 1 + \frac{6}{z} + \frac{1}{z^2} \). If we now let \( u = \frac{1}{z} \), we have the parabola \( 1 + 6u + u^2 \) which opens up and has its minimum at \( u = -3 \) with minimum value of \(-8\). Note that since \( x \) can assume any real value except 4, we know that \( z \) and \( u \) will assume all real values except zero. Thus, the minimum value of this expression is \(-8\).

16. **Solution 1**

Since the function \( g \) is linear and has positive slope, it is one-to-one and so it is invertible.

This means that \( g^{-1}(g(a)) = a \) for every real number \( a \) and \( g(g^{-1}(b)) = b \) for every real number \( b \).

Therefore, \( g(f(g^{-1}(g(a)))) = g(f(a)) \) for every real number \( a \).

This means that

\[
g(f(a)) = g(f(g^{-1}(g(a)))) \\
= 2(g(a))^2 + 16g(a) + 26 \\
= 2(2a - 4)^2 + 16(2a - 4) + 26 \\
= 2(4a^2 - 16a + 16) + 32a - 64 + 26 \\
= 8a^2 - 6
\]

Furthermore, if \( b = f(a) \), then \( g^{-1}(g(f(a))) = g^{-1}(g(b)) = b = f(a) \). Therefore,

\[
f(a) = g^{-1}(g(f(a))) = g^{-1}(8a^2 - 6)
\]

Since \( g(x) = 2x - 4 \), we have \( y = 2g^{-1}(y) - 4 \) and so \( g^{-1}(y) = \frac{1}{2}y + 2 \). Therefore,

\[
f(a) = \frac{1}{2}(8a^2 - 6) + 2 = 4a^2 - 1
\]

and so \( f(\pi) = 4\pi^2 - 1 \).

**Solution 2**

Since the function \( g \) is linear and has positive slope, it is one-to-one and so it is invertible.

To find a formula for \( g^{-1}(y) \), we start with the equation \( g(x) = 2x - 4 \), convert to \( y = 2g^{-1}(y) - 4 \) and then solve for \( g^{-1}(y) \) to obtain \( 2g^{-1}(y) = y + 4 \) and so \( g^{-1}(y) = \frac{y + 4}{2} \).

We are given that \( g(f(g^{-1}(x))) = 2x^2 + 16x + 26 \).
We can apply the function \( g^{-1} \) to both sides successively to obtain

\[
\begin{align*}
  f(g^{-1}(x)) &= g^{-1}(2x^2 + 16x + 26) \\
  f(g^{-1}(x)) &= \frac{(2x^2 + 16x + 26) + 4}{2} \quad \text{(knowing a formula for } g^{-1}) \\
  f(g^{-1}(x)) &= x^2 + 8x + 15 \\
  f\left(\frac{x + 4}{2}\right) &= x^2 + 8x + 15 \quad \text{(knowing a formula for } g^{-1}) \\
  f\left(\frac{x + 4}{2}\right) &= x^2 + 8x + 16 - 1 \\
  f\left(\frac{x + 4}{2}\right) &= (x + 4)^2 - 1
\end{align*}
\]

We want to determine the value of \( f(\pi) \).

Thus, we can replace \( \frac{x + 4}{2} \) with \( \pi \), which is equivalent to replacing \( x + 4 \) with \( 2\pi \).

Thus, \( f(\pi) = (2\pi)^2 - 1 = 4\pi^2 - 1 \).