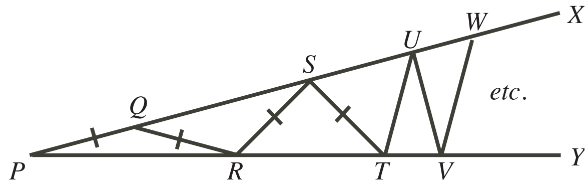
Problem of the Week
Grade 9 and 10

Isosceles Triangles Rule
Solution

Problem

In the diagram, $\angle XPY = 15^\circ$. Points $Q, R, S, T, \ldots$ alternate from one arm of the angle to the other, each point located farther away from P than the point before. If $PQ = QR = RS = \ldots$, what is the maximum number of isosceles triangles with equal sides of length PQ that can be formed?



Solution

Throughout this solution there will be references to the exterior angle theorem. The theorem states: "An exterior angle of a triangle is equal to the sum of the opposite interior angles inside the triangle." For example, $\angle SQR$ is exterior to $\triangle QPR$ and $\angle SQR = \angle QPR + \angle PRQ$.

Extend the diagram so a few more triangles are created such that $PQ = QR = RS = ST = TU = UV = \ldots$. In the solution we will show that the maximum number of isosceles triangles that can be created is 5.

In $\triangle QPR$, $\angle QPR = 15^\circ$ and $PQ = QR$. Therefore $\triangle QPR$ is isosceles and $\angle QRP = \angle QPR = 15^\circ$. $\angle SQR$ is exterior to $\triangle QPR$ so, by the exterior angle theorem for triangles, $\angle SQR = \angle QPR + \angle QRP = 15^\circ + 15^\circ = 30^\circ$.

In $\triangle RQS$, $\angle SQR = 30^\circ$ and $QR = RS$. Therefore $\triangle RQS$ is isosceles and $\angle RSQ = \angle SQR = 30^\circ$. $\angle SRT$ is exterior to $\triangle PRS$ so, by the exterior angle theorem for triangles, $\angle SRT = \angle SPR + \angle PSR = 15^\circ + 30^\circ = 45^\circ$.

In $\triangle SRT$, $\angle SRT = 45^\circ$ and $SR = ST$. Therefore $\triangle SRT$ is isosceles and $\angle STR = \angle SRT = 45^\circ$. $\angle UST$ is exterior to $\triangle PST$ so, by the exterior angle theorem for triangles, $\angle UST = \angle SPT + \angle STP = 15^\circ + 45^\circ = 60^\circ$.

In $\triangle TSU$, $\angle UST = 60^\circ$ and $ST = TU$. Therefore $\triangle TSU$ is isosceles and $\angle TUS = \angle UST = 60^\circ$. $\angle VTU$ is exterior to $\triangle PUT$ so, by the exterior angle theorem for triangles, $\angle VTU = \angle UPT + \angle PUT = 15^\circ + 60^\circ = 75^\circ$.

In $\triangle UTV$, $\angle UTV = 75^\circ$ and $TU = UV$. Therefore $\triangle UTV$ is isosceles and $\angle UVT = \angle VTU = 75^\circ$. $\angle WUV$ is exterior to $\triangle PUV$ so, by the exterior angle theorem for triangles, $\angle WUV = \angle UPV + \angle PVU = 15^\circ + 75^\circ = 90^\circ$.

In $\triangle VUW$, $\angle WUV = 90^\circ$ and $VU = VW$. Therefore $\triangle VUW$ is isosceles and $\angle VWU = \angle WUV = 90^\circ$. But this is not possible. The three angles in a triangle sum to 180° so a triangle cannot have two 90° angles and $\triangle WUV$ cannot be isosceles.

$\therefore$ **five isosceles triangles, $\triangle$QPR, $\triangle$RQS, $\triangle$SRT, $\triangle$TSU, and $\triangle$UTV, can be formed.**

A similar but more difficult problem can be found on the 2003 Cayley contest at question 25.