Problem of the Week
Grade 9 and 10

A Number in a Haystack
Solution

Problem

The digits 1, 2, 3, 4, and 5 are each used exactly once to create a five digit number $abcde$ which satisfies the following conditions: (i) the three digit number abc is divisible by 4; (ii) the three digit number bcd is divisible by 5; and (iii) the three digit number cde is divisible by 3. Find all five digit numbers formed, using each of the digits 1 to 5 exactly once, that satisfy all three conditions.

Solution

In condition (ii), bcd is divisible by 5. For a number to be divisible by 5 it must end in 0 or 5. $d = 0$ is not permitted. So $d = 5$ is the only possibility. Therefore the number looks like $abc5e$.

In condition (ii), abc is divisible by 4. For a number to be divisible by 4, its last two digits must be divisible by 4. This also means that the number is even and the last digit cannot be odd. It follows that bc must be divisible by 4 and c must be even. Therefore, $c = 2$ or $c = 4$ and the number looks like $ab25e$ or $ab45e$.

In condition (iii), cde is divisible by 3. For a number to be divisible by 3, the sum of its digits must be divisible by 3. The sum $c + d + e$ must be divisible by 3. But we know that $d = 5$ and $c = 2$ or $c = 4$ so $2 + 5 + e$ and $4 + 5 + e$ must be divisible by 3.

The expression $2 + 5 + e$ simplifies to $7 + e$. e can only be one of the remaining digits 1, 3 or 4. For $e = 1$, $7 + e = 7 + 1 = 8$ which is not divisible by 3. For $e = 3$, $7 + e = 7 + 3 = 10$ which is not divisible by 3. For $e = 4$, $7 + e = 7 + 4 = 11$ which is not divisible by 3. No value of e exists so the $ab25e$ is divisible by 3.

The other expression, $4 + 5 + e$ simplifies to $9 + e$. e can only be one of the remaining digits 1, 2 or 3. For $e = 1$, $9 + e = 9 + 1 = 10$ which is not divisible by 3. For $e = 2$, $9 + e = 9 + 2 = 11$ which is not divisible by 3. For $e = 3$, $9 + e = 9 + 3 = 12$ which is divisible by 3. So $e = 3$ is the only value of e that exists so the $ab45e$ is divisible by 3. Therefore the number looks like $ab453$.

From $ab453$ we need $ab4$, the first three digits, to be divisible by 4. In fact, we need $b4$ to be divisible by 4. There are only two possible values for b remaining, namely $b = 1$ or $b = 2$. If $b = 1$, $b4 = 14$ which is not divisible by 4. If $b = 2$, $b4 = 24$ which is divisible by 4. Therefore $b = 2$ is the only value for b. The number is now $a2453$.

The only value remaining for a is 1. Therefore the only number that satisfies all of the conditions is 12453.

There is only one number, 12453, that satisfies all of the conditions of the problem.