# Problem of the Week
## Grade 7 and 8

## Powerful Factorials At Work!
## Solution

### Problem

The product of the positive integers 1 to 3 is $3 \times 2 \times 1 = 6$ and can be written in an abbreviated form as 3!. We say 3 $factorial$. So $3! = 6$. The product of the positive integers 1 to 17 is $17 \times 16 \times 15 \times \cdots \times 3 \times 2 \times 1$ and can be written in an abbreviated form as 17!. We say 17 $factorial$. The $\cdots$ represents the product of all the missing integers between 15 and 3. In general, the product of the positive integers 1 to $n$ is $n!$. Note that $1! = 1$. Determine the units digit in the sum $1! + 2! + 3! + \ \cdots\ + 18! + 19! + 20!$.

### Solution

At first glance it may look as if there is a great deal of work to do. However by examining several factorials we will discover otherwise.

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \times 1 = 2 \\ 3! &= 3 \times 2 \times 1 = 6 \\ 4! &= 4 \times 3 \times 2 \times 1 = 24 \\ 5! &= 5 \times 4 \times 3 \times 2 \times 1 = 120 \end{aligned}$$

Now $6! = 6 \times (5 \times 4 \times 3 \times 2 \times 1) = 6 \times 5! = 6(120) = 720$.
And $7! = 7 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 7 \times 6! = 7(720) = 5040$.
An interesting observation surfaces, $8! = 8 \times 7!,\ 9! = 9 \times 8!,$ and so on.

Furthermore, 5! has a units digit of 0. Every factorial above 5! will also have a units digit of 0 since multiplying a number by another number whose units digit is zero produces a zero in the units digit of the product. So no factorials above 4! will have a units digit other than zero. Therefore, the units digit of the sum $1! + 2! + 3! + 4!\ \cdots\ + 18! + 19! + 20!$ will come from the units digit of 1!, 2!, 3! and 4!. Calculating the sum $1! + 2! + 3! + 4!$ we obtain $1 + 2 + 6 + 24 = 33$. The units digit of the required sum is the units digit of 33, namely 3.

$\therefore$ the units digit of the sum $1! + 2! + 3!\ \cdots\ + 18! + 19! + 20!$ is 3.