Problem of the Week
Grade 11 and 12

You Make a Difference
Solution

Problem

The sequence $a_1 = 2,\ a_2 = 5,\ a_3 = 12,\ \cdots,\ a_n,\ \cdots$, where a_n is the n^{th} term in the sequence, has the property that the difference between consecutive terms forms an arithmetic sequence. Find a formula for a_n in terms of n.

Three solutions are provided for this problem.

Solution 1

Let the sequence of differences be $t_1,\ t_2,\ t_3,\ \cdots,\ t_n,\ \cdots$.

Then $t_1 = a_2 - a_1 = 5 - 2 = 3$ and $t_2 = a_3 - a_2 = 12 - 5 = 7$. Since the new sequence is arithmetic, the difference between consecutive terms is constant. Therefore the constant difference is $d = t_2 - t_1 = 7 - 3 = 4$. We can now generate a few more terms in the sequence of differences by adding the constant to the previous term. $t_3 = t_2 + 4 = 7 + 4 = 11$ and $t_4 = t_3 + 4 = 11 + 4 = 15$.

Using the information we can present the information in tabular form.

Term Number n	First Sequence a_n	First Difference t_n	Second Difference (Difference of the differences)
1	2	3	4
2	5	7	4
3	12	11	4
4	23	15	
5	38		

Since the second difference is constant we can represent the general term of the first sequence with a quadratic function. Let $a_n = pn^2 + qn + r$.

For $n = 1$, $a_1 = 2 = p(1)^2 + q(1) + r$. $\therefore\ p + q + r = 2$. (1)
For $n = 2$, $a_2 = 5 = p(2)^2 + q(2) + r$. $\therefore 4p + 2q + r = 5$. (2)
For $n = 3$, $a_3 = 12 = p(3)^2 + q(3) + r$. $\therefore 9p + 3q + r = 12$. (3)

Subtracting (1) from (2), $3p + q = 3$. (4)
Subtracting (2) from (3), $5p + q = 7$. (5)

Subtracting (4) from (5), $2p = 4$ and $p = 2$ follows.
Substituting $p = 2$ into (4), $3(2) + q = 3$ and $q = -3$ follows.

Substituting $p = 2,\ q = -3$ into (1), $2 - 3 + r = 2$ and $r = 3$ follows.

$\therefore a_n = 2n^2 - 3n + 3$ is the general term of the given sequence in terms of n.

The second solution approaches the problem in a more general sense.

Problem

The sequence $a_1 = 2$, $a_2 = 5$, $a_3 = 12$, $\cdots$, a_n, $\cdots$, where a_n is the n^{th} term in the sequence, has the property that the difference between consecutive terms forms an arithmetic sequence. Find a formula for a_n in terms of n.

Solution 2

Let the sequence of differences be $t_1,\ t_2,\ t_3,\ \cdots\ ,\ t_n,\ \cdots$. Then $t_1 = a_2 - a_1 = 3$ and $t_2 = a_3 - a_2 = 7$. Since the new sequence is arithmetic, the constant difference is $d = 7 - 3 = 4$. We can generate more terms: $t_3 = 11,\ t_4 = 15,\ t_5 = 19,\ \cdots$.

Each term in the new sequence is the difference between consecutive terms of the original sequence.

$$
\begin{array}{rcccl}
 & \cancel{a_2} & - & a_1 & = & t_1 \\
 & \cancel{a_3} & - & \cancel{a_2} & = & t_2 \\
 & \cancel{a_4} & - & \cancel{a_3} & = & t_3 \\
 & \vdots & & \vdots & & \vdots \\
 & \cancel{a_{n-2}} & - & \cancel{a_{n-3}} & = & t_{n-3} \\
 & \cancel{a_{n-1}} & - & \cancel{a_{n-2}} & = & t_{n-2} \\
 & a_n & - & \cancel{a_{n-1}} & = & t_{n-1} \\
\hline
\text{Adding,} & a_n & - & a_1 & = & t_1 + t_2 + t_3 + \ \cdots\ + t_{n-2} + t_{n-1} = S_{n-1}. \quad (1)
\end{array}
$$

To find the sum $S_{n-1} = t_1 + t_2 + t_3 + \ \cdots\ + t_{n-2} + t_{n-1}$ we can use the formula $S_n = \dfrac{n}{2}[2a + (n-1)d]$ with $a = 3$ and $d = 4$ for $(n-1)$ terms. So

$$
\begin{aligned}
S_{n-1} &= t_1 + t_2 + t_3 + \ \cdots\ + t_{n-2} + t_{n-1} \\
&= \frac{n-1}{2}[2a + ((n-1)-1)d] \\
&= \frac{n-1}{2}[2(3) + ((n-1)-1)(4)] \\
&= \frac{n-1}{2}[6 + (n-2)(4)] \\
&= \frac{n-1}{2}[6 + 4n - 8] \\
&= \frac{n-1}{2}[4n - 2] \\
&= (n-1)(2n-1)
\end{aligned}
$$

From (1) above, $a_n - a_1 = S_{n-1}$ so $a_n = S_{n-1} + a_1 = (n-1)(2n-1) + 2 = 2n^2 - 3n + 3$.

$\therefore\ a_n = 2n^2 - 3n + 3$ is the general term of the given sequence in terms of n.

Solution 3 approaches the problem by examining what the n^{th} term, a_n, looks like.

Problem

The sequence $a_1 = 2,\ a_2 = 5,\ a_3 = 12,\ \cdots,\ a_n,\ \cdots$, where a_n is the n^{th} term in the sequence, has the property that the difference between consecutive terms forms an arithmetic sequence. Find a formula for a_n in terms of n.

Solution 3

Let the sequence of differences be $t_1,\ t_2,\ t_3,\ \cdots,\ t_n,\ \cdots$. Then $t_1 = a_2 - a_1 = 3$ and $t_2 = a_3 - a_2 = 7$. Since the new sequence is arithmetic, the constant difference is $d = 7 - 3 = 4$. Using the formula for the general term of an arithmetic sequence, $t_n = a + (n-1)d$, the general term of the sequence of differences is $t_n = (3) + (n-1)(4) = 3 + 4n - 4 = 4n - 1$.

To generate the original sequence we start with the first term and add more terms from the arithmetic sequence of differences. For example,

$$a_1 = 3,$$
$$a_2 = 2 + t_1 = 2 + [4(1) - 1] = 2 + 3 = 5, \text{and}$$
$$a_3 = 2 + t_1 + t_2 = 2 + [4(1) - 1] + [4(2) - 1] = 2 + 3 + 7 = 12$$

$$\begin{aligned}
\text{So, } a_n &= 2 + t_1 + t_2 + t_3 + \cdots + t_{n-1} \\
&= 2 + [4(1) - 1] + [4(2) - 1] + [4(3) - 1] + \cdots + [4(n-1) - 1] \\
&= 2 + [4(1) + 4(2) + 4(3) + \cdots + 4(n-1)] + (n-1)(-1) \\
&= 2 + 4[1 + 2 + 3 + \cdots + (n-1)] - n + 1 \\
&= 3 - n + 4\left[\frac{(n-1)(n)}{2}\right] \qquad (1) \\
&= 3 - n + 2(n^2 - n) \\
&= 3 - n + 2n^2 - 2n \\
&= 2n^2 - 3n + 3
\end{aligned}$$

In (1) above, $1 + 2 + 3 + \cdots + (n-1)$ is an arithmetic sequence with $t_1 = 1$, $t_{n-1} = (n-1)$ and the number of terms $(n-1)$ terms. Using the formula for the sum of the terms of an arithmetic sequence, $S_n = n\left(\frac{t_1 + t_n}{2}\right)$, we obtain

$$S_{n-1} = (n-1)\left(\frac{1 + (n-1)}{2}\right) = (n-1)\left(\frac{n}{2}\right) = \frac{(n-1)(n)}{2}$$

$\therefore a_n = 2n^2 - 3n + 3$ is the general term of the given sequence in terms of n.