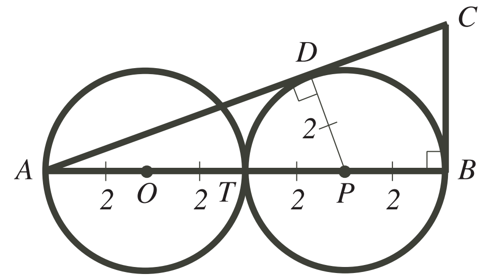
Problem of the Week
Grade 11 and 12

A Touchy Triangle
Solution

Problem

Two circles with centres O and P, each with a radius of 2, are tangent to each other. A straight line is drawn through O and P meeting the circles at A and B. Two other sides of $\triangle ABC$ are drawn such that side AC is tangent to the circle with centre P at D and side CB is tangent to the circle with centre P at B. Determine the length of BC.



Solution

Let T be the point of tangency of the two circles. Then $AO = OT = TP = PB = 2$. Since AC is tangent to the circle with centre P at D, $CD \perp PD$. Since CB is tangent to the circle with centre P at B, $CB \perp PB$. Since PD and PB are radii of the circle with centre P, $PD = PB = 2$. This information has been added to the diagram.

$\triangle ADP$ is right angled at D since $AD \perp PD$. So $AD^2 = AP^2 - PD^2 = 6^2 - 2^2 = 36 - 4 = 32$. Since $AD > 0$, $AD = \sqrt{32} = 4\sqrt{2}$.

At this point there are many ways to find the length of CB. An interesting (and possibly different) way to find CB is to use basic trigonometry in the right triangles. In $\triangle APD$, $\tan(A) = \frac{PD}{AD}$ and in $\triangle ACB$, $\tan(A) = \frac{CB}{AB}$.

$$\begin{aligned} \therefore \frac{PD}{AD} &= \frac{CB}{AB} \\ \frac{2}{4\sqrt{2}} &= \frac{CB}{8} \\ CB &= \frac{16}{4\sqrt{2}} = \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2} \end{aligned}$$

Therefore the length of CB is $2\sqrt{2}$.

There are many other solutions to this problem. The solver can use Pythagoras' Theorem in $\triangle ABC$. Another solution uses similar triangles $\triangle APD$ and $\triangle ACB$.