Problem of the Week
Grade 11 and 12

Trapped!
Solution

Problem

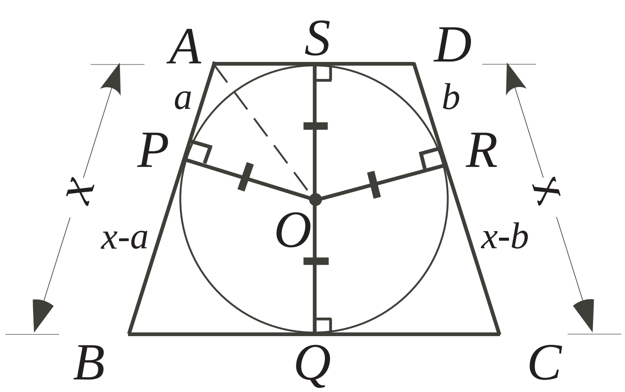
Isosceles trapezoid $ABCD$ has parallel sides AD and BC and sides $AB = DC = x$. The area of the trapezoid is 510 cm^2. A circle with centre O and radius 10 cm is contained inside the trapezoid so that it is tangent to each of the four sides of the trapezoid. Determine the length of x.

For purposes of this problem accept the fact that a line drawn from the centre of a circle to a point of tangency is perpendicular to the tangent.

Solution

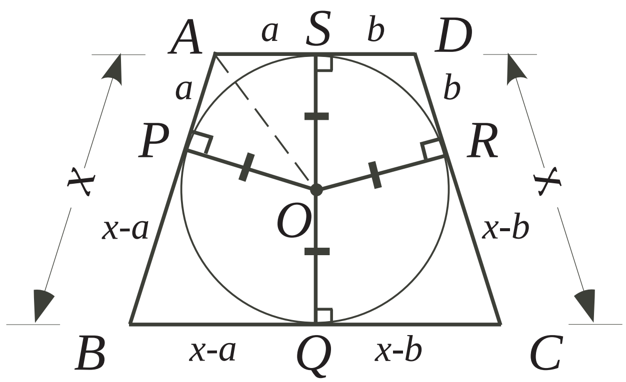
Draw radii OP, OQ, OR, and OS to sides AB, BC, CD, and DA, respectively. Then $OP = OQ = OR = OS = 10$ since they are each radii of the circle. Also, since each line drawn from the centre of the circle to a point of tangency is perpendicular to the tangent,
$\angle OPA = \angle OQB = \angle ORD = \angle OSA = 90^\circ$.

Let $AP = a$ and $DR = b$. Therefore, $PB = x - a$ and $RC = x - b$. The following diagram shows all of the given and found information.



Join A to O forming two right triangles, $\triangle APO$ and $\triangle ASO$. Using the Pythagorean Theorem, $AP^2 = AO^2 - OP^2$ and $AS^2 = AO^2 - OS^2$. But $OP = OS$ since they are both radii. So the two expressions are equal and $AS = AP = a$ follows.

Using exactly the same reasoning that was used to show $AP = AS = a$, we can show $DR = DS = b$, $BP = BQ = x - a$ and $CR = CQ = x - b$. This new information has been added to the diagram below.



We can now find the area of the trapezoid. To find the area of a trapezoid, we multiply the perpendicular distance between the two parallel sides by the sum of the lengths of the two parallel sides and then divide the result by 2.

$$\begin{aligned}
\text{Area Trapezoid } ABCD &= SQ \times (AD + BC) \div 2 \\
510 &= (SO + OQ) \times ((AS + SD) + (BQ + QC)) \div 2 \\
510 &= (10 + 10) \times ((a + b) + (x - a + x - b)) \div 2 \\
510 &= (20) \times (2x) \div 2 \\
510 &= 20x \\
25.5 &= x
\end{aligned}$$

Therefore, the length of x is 25.5 cm.

At the end of the statement of the problem, the following fact was assumed: *a line drawn from the centre of a circle to a point of tangency is perpendicular to the tangent*. As an extension to this problem the solver may wish to try to prove this fact.