Problem of the Week
Grade 11 and 12

A Problem For The Ages
Solution

Problem

A four-digit number which is a perfect square is created by writing Jennifer's age in years followed by John's age in years. Similarly, in 31 years, their ages in the same order will again form a four-digit perfect square. How old are Jennifer and John today?

Solution

Both Jennifer's age and John's age must be two digit numbers. If Jennifer's age is a one-digit number, John's age would have to be a three-digit number to create the four-digit perfect square. But in 31 years, Jennifer's age would then be a two-digit number resulting in at least a five-digit number when their ages are used to form the second perfect square. A similar argument could be presented if John's age is a one-digit number. Therefore, both Jennifer and John have ages that are each two-digit numbers.

Let Jennifer's present age be x and John's present age be y. Then $100x + y$ is the four digit number created by writing Jennifer's age followed by John's age. But $100x + y$ is a perfect square so let $100x + y = k^2$, for some positive integer k.

In 31 years, Jennifer will be $(x + 31)$ and John will be $(y + 31)$. The new number created by writing Jennifer's age followed by John's age is $100(x + 31) + (y + 31)$. This new four-digit number is also a perfect square. So $100(x + 31) + (y + 31) = m^2$, for some positive integer $m,\ m > k$. This simplifies as follows:

$$100x + 3100 + y + 31 = m^2$$
$$100x + y + 3131 = m^2 \quad (1)$$

From our work above, we also have $100x + y = k^2$. Substituting this into (1) we get $k^2 + 3131 = m^2$ or $3131 = m^2 - k^2$. $m^2 - k^2$ is a difference of squares, so $m^2 - k^2 = (m + k)(m - k) = 3131$.

Since m and k are positive integers, $m + k$ is positive and $m + k > m - k$. $m - k$ must also be positive since $(m + k)(m - k) = 3131$. So we are looking for two positive numbers that multiply to 3131. There are two possibilities, 3131×1 or 101×31.

First we will examine $(m + k)(m - k) = 3131 \times 1$. From this we obtain two equations in two unknowns, namely $m + k = 3131$ and $m - k = 1$. Subtracting the two equations gives $2k = 3130$ or $k = 1565$. Then $k^2 = 1565^2 = 2449225$. This is not a four-digit number so 3131×1 is not an admissible factorization of 3131.

Next we examine $(m + k)(m - k) = 101 \times 31$. This leads to $m + k = 101$ and $m - k = 31$. Subtracting the two equations we get $2k = 70$ or $k = 35$. Then $100x + y = k^2 = 1225$. Therefore, $x = 12$ and $y = 25$ since 1225 is the four-digit number formed by writing Jennifer's age, x, followed by John's age, y.

$\therefore$ **today Jennifer is 12 and John is 25.**