Problem of the Week
Grade 11 and 12

Powerfully Perfect Squares
Solution

Problem

The prime factorization of 20 is $2^2 \times 5$. The divisors of 20 are $2^0 5^0 = 1,\ 2^0 5^1 = 5,\ 2^1 5^0 = 2,\ 2^1 5^1 = 10,\ 2^2 5^0 = 4,$ and $2^2 5^1 = 20$. The number 20 has 6 divisors. Two of the divisors, 1 and 4, are perfect squares. How many divisors of 2012^{2012} are perfect squares?

Solution

First, let's look at the prime factorization of some perfect squares.
$9 = 3^2,\ 16 = 2^4,\ 36 = 2^2 \times 3^2$, and $144 = 2^4 \times 3^2$ are the prime factorizations of four perfect squares. Note that the exponent on each of the prime factors is even. For some integer a, if m is an even integer greater than or equal to zero then a^m is a perfect square.

Now $2012^{2012} = (2^2 \times 503)^{2012} = (2^2)^{2012} \times 503^{2012} = 2^{4024} \times 503^{2012}$. All divisors of 2012^{2012} will be of the form $2^k \times 503^n$, $0 \leq k \leq 4024$, $0 \leq n \leq 2012$ where k and n are both integers. For 2^k to be a perfect square, k must be an even integer such that $0 \leq k \leq 4024$. There are $4024 \div 2 = 2012$ even numbers from 1 to 4024. The number 0 is also even so there are 2013 values of k such that 2^k is a perfect square.

For 503^n to be a perfect square, n must be an even integer such that $0 \leq n \leq 2012$. There are 1006 even numbers from 1 to 2012. Zero is also even so there are 1007 values of n such that 503^n is a perfect square.
For each of the 2013 values of k, there are 1007 values of n so there are $2013 \times 1007 = 2\ 027\ 091$ perfect square divisors of 2012^{2012}.

$\therefore\ 2012^{2012}$ has 2 027 091 perfect square divisors.

(For a smaller number like 12^6 we could also determine the number of perfect square divisors using the above approach. We could verify that the approach is valid by physically listing all of the divisors and then counting the number that are perfect squares. The number 12^6 has "only" 91 divisors, 28 of which are perfect squares. It is not practical to list all the divisors of 2012^{2012} and then count the perfect squares. But our approach allows us to still count the perfect square divisors.)