Problem of the Week
Problem E and Solution
No Bills

Problem

Given an unlimited supply of Loonies, Toonies and Quarters, in how many different ways is it possible to make a total of exactly \$100?

Solution

We will break into cases based on how many \$2 coins we can have. For each case, we will count the number of possibilities for the number of \$1 and 25¢ coins.

The maximum number of \$2 coins we can have is 50, since $\$2 \times 50 = \$100$. If we have 50 \$2 coins, then we do not need any \$1 or 25¢ coins. Therefore, there is only one way to make a total of \$100 if there are 50 \$2 coins.

Suppose we have 49 \$2 coins. Since $\$2 \times 49 = \$98$, to reach a total of \$100, we would need two \$1 and no 25¢ coins, or one \$1 and four 25¢ coins, or no \$1 and eight 25¢ coins. Therefore, there are 3 different ways to make a total of \$100 if there are 49 \$2 coins.

Suppose we have 48 \$2 coins. Since $\$2 \times 48 = \$96$, to reach a total of \$100, we would need four \$1 and no 25¢ coins, or three \$1 and four 25¢ coins, or two \$1 and eight 25¢ coins, or one \$1 and twelve 25¢ coins, or no \$1 and sixteen 25¢ coins. Therefore, there are 5 different ways to make a total of \$100 if there are 48 \$2 coins.

We start to see a pattern. When we reduce the number of \$2 coins by one, the number of possible combinations using that many \$2 coins increases by 2. This is because we can increase the number of \$1 coins by 1 or 2, so there are two new possibilities.

When there are 47 \$2 coins, there are 7 possible ways to make a total of \$100.
When there are 46 \$2 coins, there are 9 possible ways to make a total of \$100, and so on.

When there is one \$2 coin, there are 99 different ways to make up the difference of \$98 (you can use 0 to 98 \$1 coins).
When there are no \$2 coins, there are 101 different ways to get a total of \$100 (you can use 0 to 100 \$1 coins). Therefore, the number of different ways to make a total of exactly \$100 is

$$
\begin{aligned}
&1+3+5+7+9+\ldots+99+101\\
&=1+2+3+4+5+6+7+8+9+\ldots+98+99+100+101-(2+4+6+8+\ldots+98+100)\\
&\qquad\qquad\text{(add and subtract the even numbers from 2 to 100)}\\
&=(1+2+3+\ldots+100+101)-2(1+2+3+4+\ldots+50)\quad\text{(factor out a 2 from the even numbers)}\\
&=\frac{101(102)}{2}-2\left(\frac{50(51)}{2}\right)\qquad\text{(using the formula for the sum of the first } n \text{ positive integers)}\\
&=101(51)-50(51)\\
&=2601
\end{aligned}
$$

Therefore, there are 2601 different combinations of coins that can be used to make \$100.

Extending the ideas

Let's look at the end of the previous computation another way.

$$\begin{aligned}
&1+3+5+7+9+\ldots+99+101 \\
&= \frac{101(102)}{2} - 2\left(\frac{50(51)}{2}\right) && \text{(using the formula for the sum of the first } n \text{ positive integers)} \\
&= 101(51) - 50(51) && \text{(simplify)} \\
&= 51(101-50) && \text{(common factor 51 from both terms)} \\
&= 51(51) && \text{(simplify)} \\
&= 51^2
\end{aligned}$$

How many odd integers are in the list 1 to 101?

From 1 to 101, there are 101 integers.
This list contains the even integers, 2 to 100, 50 in total.
Therefore, there are $101 - 50 = 51$ odd integers from 1 to 101.

Is it a coincidence that the sum of the first 51 odd positive integers is 51^2? Is the sum of the first 1000 odd positive integers 1000^2? Is the sum of the first n odd positive integers n^2?

We will develop a formula for the sum of the first n odd positive integers.

We saw in the problem statement that the sum of the first n positive integers is

$$1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

Every odd positive integer can be written in the form $2n-1$, where n is an integer ≥ 1. When $n = 1$, $2n-1 = 2(1)-1 = 1$; when $n = 2$, $2n-1 = 2(2)-1 = 3$, and so on. So the 51st odd positive integer is $2(51)-1 = 101$, as we determined above. The nth odd positive integer is $2n-1$. Let's consider the sum of the first n odd positive integers. That is,

$$1+3+5+7+\ldots+(2n-3)+(2n-1)$$

$$\begin{aligned}
&1+3+5+7+\ldots+(2n-3)+(2n-1) \\
&= 1+2+3+4+5+\ldots+(2n-3)+(2n-2)+(2n-1)+2n-(2+4+6+\ldots+(2n-2)+2n) \\
& \qquad \text{(add and subtract the even numbers from 2 to } 2n\text{)} \\
&= (1+2+3+4+\ldots+2n)-(2+4+6+8+\ldots+(2n-2)+2n) \\
&= (1+2+3+4+\ldots+2n)-2(1+2+3+\ldots+n) && \text{(factor out a 2 from the even numbers)} \\
&= \frac{2n(2n+1)}{2} - 2\left(\frac{n(n+1)}{2}\right) && \text{(using the formula for the sum of the first } n \text{ integers)} \\
&= n(2n+1)-n(n+1) && \text{(simplify)} \\
&= 2n^2+n-n^2-n && \text{(simplify)} \\
&= n^2
\end{aligned}$$

Therefore, the sum of the first n odd positive integers is n^2.

For further thought.

Can you develop a formula for the sum of the first n even positive integers?