Problem of the Week
Problem E and Solution
No Repeat

Problem
When \( \frac{1}{50^{2018}} \) is written as a decimal, it terminates.

What is the last non-zero digit in the decimal representation of \( \frac{1}{50^{2018}} \)?

Solution
Our first instinct might be to use our calculator to get an idea of how the last digit behaves for the first few powers of \( \frac{1}{50} \). Most calculators let us down too quickly.

Notice that
\[
\frac{1}{50^{2018}} = \left( \frac{1}{50} \right)^{2018} = \left( \frac{1}{100} \times 2 \right)^{2018} = (0.01 \times 2)^{2018} = (0.01)^{2018} \times 2^{2018}
\]

The last non-zero digit in the decimal representation of \( \frac{1}{50^{2018}} \) will therefore be the last non-zero digit in the decimal representation of \( (0.01)^{2018} \) multiplied by the last digit of \( 2^{2018} \).

Since the last non-zero digit in the decimal representation of \( (0.01)^{2018} \) is 1, the last non-zero digit in the decimal representation of \( \frac{1}{50^{2018}} \) will therefore be the last digit of \( 2^{2018} \).

We now examine the last digit of various powers of 2:

\[
\begin{align*}
2^1 &= 2 \\
2^2 &= 4 \\
2^3 &= 8 \\
2^4 &= 16 \\
2^5 &= 32 \\
2^6 &= 64 \\
2^7 &= 128 \\
2^8 &= 256
\end{align*}
\]

Notice that the last digit repeats every four powers of 2. This pattern continues. \( 2^9 \) ends with a 2, \( 2^{10} \) ends with a 4, \( 2^{11} \) ends with an 8, \( 2^{12} \) ends with a 6, and so on.

We need to determine the number of complete cycles in 2018.

\[
\frac{2018}{4} = 504 \frac{1}{2}
\]

Therefore, there are 504 complete cycles. Since \( 504 \times 4 = 2016 \), this means \( 2^{2016} \) ends with a 6, \( 2^{2017} \) ends with a 2, and \( 2^{2018} \) ends with a 4.

Since \( 2^{2018} \) ends with a 4, \( \frac{1}{50^{2018}} \) also ends with a 4.