Problem of the Week
Problem E and Solution
Piecing It Together

Problem

A positive integer is written on the back of each of three puzzle pieces. The numbers are not necessarily different but the total of the three numbers is 14. Each puzzle piece is placed on a table so that the number cannot be seen. Alpha, Beta and Gamma each select one of the pieces, being careful not to let the other two see the number that is printed on the piece. Alpha says, after looking at his puzzle piece, "I know that Beta and Gamma have different numbers." Beta then says, "I already knew that all three numbers were different." At this point, Gamma confidently exclaims, "I now know what all three of the numbers are!" What were the numbers and who had which number?

Solution

We want to find the single solution to the problem $x + y + z = 14$ that satisfies the statements offered by Alpha, Beta and Gamma. It turns out that there are 78 different possible sums of three positive integers totalling 14. We could list all of the possible solutions and then proceed through the statements until we determine the required solution. Our approach will be far less exhausting. At the end of the solution, a justification of the existence of 78 possible positive integral solutions to the equation $x + y + z = 14$ will be provided.

The sum of three numbers is 14, an even number. To generate an even sum, the three numbers must all be even or one of the numbers must be even and the other two numbers must be odd.

Alpha says, after looking at his puzzle piece, "I know that Beta and Gamma have different numbers." How can Alpha KNOW? If his number is even then Beta and Gamma could both have even numbers or both have odd numbers to generate the sum 14. For example, if Alpha had the number 6, Beta could have 6 and Gamma could have 2 or Beta could have 4 and Gamma could have 4. If his number was even, Alpha would not KNOW that the other two numbers were different. Therefore, Alpha must have an odd number, one of the others has an odd number and the other has an even number.

Beta then says, "I already knew that all three numbers were different." Using the same logic as before, since Beta knows Alpha and Gamma have different numbers, Beta must have an odd number (and thus Gamma must have the even number). But how does Beta KNOW that all three numbers are different?

If Beta has a 1, 3 or 5, Alpha could have the same number. Beta would not know.

If Beta has a 7, 9, 11 or 13, Alpha could not have the same number in order for the three numbers to sum to 14. Furthermore, if Beta has a 7, Alpha must have a 5 or lower. If Beta has a 9, Alpha must have a 3 or lower. If Beta has an 11, Alpha must have a 1. Beta cannot have a 13 in order for the three numbers to sum to 14.

At this point our list of possible solutions has dropped from 78 to 6.

Beta	Alpha	Gamma
7	1	6
7	3	4
7	5	2
9	1	4
9	3	2
11	1	2

How does Gamma conclude, "I now know what all three of the numbers are!"?

If Gamma has a 4, Beta could have a 7 and Alpha could have a 3 or Beta could have a 9 and Alpha could have a 1. In this case, Gamma would not know.

If Gamma has a 2, Beta could have a 7 and Alpha could have a 5 or Beta could have a 9 and Alpha could have a 3 or Beta could have an 11 and Alpha could have a 1. In this case, Gamma would certainly not know.

However, if Gamma has a 6, then Beta must have a 7 and Alpha must have a 1. This is the only possibility in which Gamma's statement is true.

Therefore, Alpha has a 1, Beta has a 7, and Gamma has a 6.

NOTE: Here is the list of the 78 solutions to the equation $x + y + z = 14$ where $x,\ y,\ z$ are positive integers.

x	y	z	# of possibilities
1	1	12	
1	2	11	
1	3	10	12
$\vdots$	$\vdots$	$\vdots$	
1	12	1	
2	1	11	
2	2	10	
2	3	9	11
$\vdots$	$\vdots$	$\vdots$	
2	11	1	
3	1	10	
3	2	9	
3	3	8	10
$\vdots$	$\vdots$	$\vdots$	
3	10	1	
$\vdots$	$\vdots$	$\vdots$	
10	1	3	
10	2	2	3
10	3	1	
11	1	2	
11	2	1	2
12	1	1	1