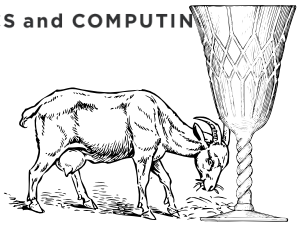
Problem of the Week
Problem E and Solution
A Certain Star Would Love This



Problem

Billy and Crystal each have a bag containing 9 balls. The balls in each bag are numbered from 1 to 9. Billy and Crystal each remove one ball from their own bag. Let b be the sum of the numbers on the 8 balls remaining in Billy's bag. Let c be the sum of the numbers on the 8 balls remaining in Crystal's bag. Determine the probability that b and c differ by a multiple of 4.

Solution

Solution 1

In order to determine the probability, we must determine the number of ways b and c can differ by a multiple of 4 and divide by the total number of ways one ball can be removed from Billy's bag and one ball can be removed from Crystal's bag.

First, let's determine the total number of ways one ball can be removed from Billy's bag and one ball can be removed from Crystal's bag. There are 9 balls in Billy's bag that can be chosen and 9 balls in Crystal's bag that can be chosen. Therefore, there are $9 \times 9 = 81$ possibilities.

The sum of the balls originally in each bag is $1+2+3+4+5+6+7+8+9 = \frac{9(10)}{2} = 45$. Since the balls are numbered from 1 to 9, when one ball is removed, the remaining sum will be from 36 to 44.

If Billy selected the ball numbered 1 from his bag, the sum of the numbers on the balls remaining in his bag would be $45 - 1 = 44$, which is b. If Crystal removed the ball with the same number from her bag, then $c = 44$ as well. Then $b - c = 0$, a multiple of 4. Crystal could also remove 5 or 9. Then $c = 45 - 5 = 40$ or $c = 45 - 9 = 36$, and $b - c = 4$ or $b - c = 8$, both of which are multiples of 4.

If Crystal removes the same number as Billy or the same number as Billy increased or decreased by some multiple of 4, such that the resulting number is a positive integer less than or equal to 9, then $b - c$ will be a multiple of 4. The table summarizes the possibilities.

Billy's Choice m	Sum of Numbers on Remaining Balls $b = 45 - m$	Crystal's Choice n	Sum of Numbers on Remaining Balls $c = 45 - n$	Difference $b - c$	Number of Selections Divisible by 4
1	44	1 or 5 or 9	44 or 40 or 36	0 or 4 or 8	3
2	43	2 or 6	43 or 39	0 or 4	2
3	42	3 or 7	42 or 38	0 or 4	2
4	41	4 or 8	41 or 37	0 or 4	2
5	40	1 or 5 or 9	44 or 40 or 36	−4 or 0 or 4	3
6	39	2 or 6	43 or 39	−4 or 0	2
7	38	3 or 7	42 or 38	−4 or 0	2
8	37	4 or 8	41 or 37	−4 or 0	2
9	36	1 or 5 or 9	44 or 40 or 36	−8 or −4 or 0	3

We find that there are $3+2+2+2+3+2+2+2+3 = 21$ different ways to draw the balls so that b and c differ by a multiple of 4.

Therefore, the probability that b and c differ by a multiple of 4 is $\frac{21}{81} = \frac{7}{27}$.

Solution 2

In Solution 1, we determined that there were 81 possibilities for selections.

Now let's determine how many of these possibilities result in the remaining sums differing by a multiple of 4.

The sum of the balls originally in each bag is $1+2+3+4+5+6+7+8+9 = \frac{9(10)}{2} = 45$. Since the balls are numbered from 1 to 9, when one ball is removed, the remaining sum will be from 36 to 44. Let's systematically look at the possible selections.

- Billy removes ball 1. Then the sum of the remaining balls in Billy's bag is $b = 45 - 1 = 44$. The sum of the balls remaining in Crystal's bag, c, must satisfy $36 \leq c \leq 44$ and $44 - c$ is a multiple of 4. There are three possibilities for c: 44, 40, 36. These values of c correspond to Crystal removing balls $1, 5$ and 9, respectively.
- Billy removes ball 2. Then the sum of the remaining balls in Billy's bag is $b = 45 - 2 = 43$. The sum of the balls remaining in Crystal's bag, c, must satisfy $36 \leq c \leq 44$ and $43 - c$ is a multiple of 4. There are two possibilities for c: 43, 39. These values of c correspond to Crystal removing balls 2 and 6, respectively.
- Billy removes ball 3. Then the sum of the remaining balls in Billy's bag is $b = 45 - 3 = 42$. The sum of the balls remaining in Crystal's bag, c, must satisfy $36 \leq c \leq 44$ and $42 - c$ is a multiple of 4. There are two possibilities for c: 42, 38. These values of c correspond to Crystal removing balls 3 and 7, respectively.
- Billy removes ball 4. Then the sum of the remaining balls in Billy's bag is $b = 45 - 4 = 41$. The sum of the balls remaining in Crystal's bag, c, must satisfy $36 \leq c \leq 44$ and $41 - c$ is a multiple of 4. There are two possibilities for c: 41, 37. These values of c correspond to Crystal removing balls 4 and 8, respectively.
- Billy removes ball 5. Then the sum of the remaining balls in Billy's bag is $b = 45 - 5 = 40$. The sum of the balls remaining in Crystal's bag, c, must satisfy $36 \leq c \leq 44$ and $40 - c$ is a multiple of 4. There are three possibilities for c: 44, 40, 36. These values of c correspond to Crystal removing balls $1, 5$ and 9, respectively.
- Billy removes ball 6. Then the sum of the remaining balls in Billy's bag is $b = 45 - 6 = 39$. The sum of the balls remaining in Crystal's bag, c, must satisfy $36 \leq c \leq 44$ and $39 - c$ is a multiple of 4. There are two possibilities for c: 43, 39. These values of c correspond to Crystal removing balls 2 and 6, respectively.
- Billy removes ball 7. Then the sum of the remaining balls in Billy's bag is $b = 45 - 7 = 38$. The sum of the balls remaining in Crystal's bag, c, must satisfy $36 \leq c \leq 44$ and $38 - c$ is a multiple of 4. There are two possibilities for c: 42, 38. These values of c correspond to Crystal removing balls 3 and 7, respectively.
- Billy removes ball 8. Then the sum of the remaining balls in Billy's bag is $b = 45 - 8 = 37$. The sum of the balls remaining in Crystal's bag, c, must satisfy $36 \leq c \leq 44$ and $37 - c$ is a multiple of 4. There are two possibilities for c: 41, 37. These values of c correspond to Crystal removing balls 4 and 8, respectively.
- Billy removes ball 9. Then the sum of the remaining balls in Billy's bag is $b = 45 - 9 = 36$. The sum of the balls remaining in Crystal's bag, c, must satisfy $36 \leq c \leq 44$ and $36 - c$ is a multiple of 4. There are three possibilities for c: 44, 40, 36. These values of c correspond to Crystal removing balls $1, 5$ and 9, respectively.

We find that there are $3+2+2+2+3+2+2+2+3 = 21$ different ways to draw the balls so that b and c differ by a multiple of 4.

Therefore, the probability that b and c differ by a multiple of 4 is $\frac{21}{81} = \frac{7}{27}$.