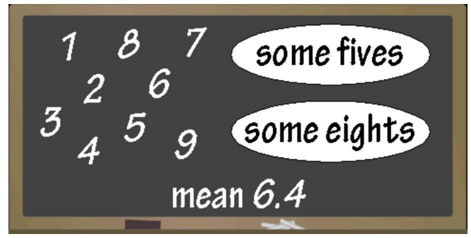
Problem of the Week

Problem E and Solution

Some of These and Some of Those

Problem



One day on the board, Ms. Math wrote the digits 1 to 9. She then wrote a certain number of fives and also a number of eights. The number of fives and the number of eights were not necessarily the same. The mean (average) of all the numbers on the board is 6.4. Determine the smallest number of numbers that can be on the board.

Solution

After writing the numbers 1 to 9 on the board, write m additional fives and n additional eights on the board. Both m and n are positive integers.

There are a total of $(9+m+n)$ numbers on the board. The sum of the numbers on the board is $(1+2+3+\cdots+7+8+9+5m+8n)$ which simplifies to $(45+5m+8n)$.

To calculate an average, determine the sum of the numbers and divide by the number of numbers.

$$\begin{array}{rrcl}
\text{Since the average is 6.4,} & \dfrac{45+5m+8n}{9+m+n} & = & 6.4 \\
\text{After multiplying both sides by 10} & \dfrac{450+50m+80n}{9+m+n} & = & 64 \\
\text{Multiplying both sides by } (9+m+n) & 450+50m+80n & = & 576+64m+64n \\
\text{Rearranging} & 16n-14m & = & 126 \\
\text{Dividing both sides by 2 and rearranging} & 8n & = & 63+7m \\
\text{Dividing both sides by 8} & n & = & \dfrac{7(9+m)}{8}
\end{array}$$

Since m and n are positive integers, $7(9+m)$ must be divisible by 8. But 8 does not divide 7 since the two numbers are relatively prime. So $(9+m)$ must be a multiple of 8.

Since m is a positive integer, $9+m$ is greater than 9. The smallest multiple of 8 which is also larger than 9 is 16. Therefore, $9+m=16$ and $m=7$.

After substituting $m=7$ into $n=\frac{7(9+m)}{8}$, we obtain $n=14$. In addition to the numbers 1 to 9, the smallest number of fives and eights that could have been written on the board would have been 7 and 14, respectively.

The smallest number of numbers that could have originally been written on the board was $9+7+14=30$. It is left as an exercise to the solver to verify that this produces the correct average. As an extension, determine the largest total number of numbers less than 1000 that could have been written on the board so that the mean is 6.4.