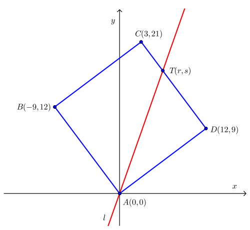


Problem of the Week

Problem E and Solution

Another Point of Division

Problem

A square has coordinates $A(0,0)$, $B(-9,12)$, $C(3,21)$ and $D(12,9)$. The line l passes through A and intersects CD at point $T(r,s)$ splitting the square so that the area of square $ABCD$ is three times the area of $\triangle ATD$. Determine the equation of line l.

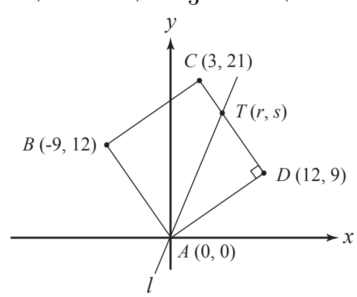
Solution

Both solutions start by finding the area of square $ABCD$, the area of $\triangle ATD$, the length AD and the length TD. We present the common start to both solutions at this point.

Using the distance formula, $AD = \sqrt{(9-0)^2+(12-0)^2} = \sqrt{81+144} = \sqrt{225} = 15$, since $AD > 0$. Therefore, the area of square $ABCD = 15^2 = 225$.

Since the area of square $ABCD$ is three times the area of $\triangle ATD$, area($\triangle ATD$) $= \frac{1}{3}\times$area(square $ABCD$) $= \frac{1}{3}(225) = 75$.

Since $ABCD$ is a square, $\angle ADC = 90^\circ$. Consider $\triangle ATD$. This triangle is a right-triangle with base $AD = 15$ and height TD. Using the formula area $= \frac{\text{base}\times\text{height}}{2}$,



$$\begin{aligned} \text{area}(\triangle ATD) &= \frac{AD \times TD}{2} \\ 75 &= \frac{15 \times TD}{2} \\ \therefore\ TD &= 10 \end{aligned}$$

Solution 1

We now calculate the equation of the line that the segment CD lies on.
Since D has coordinates $(12,9)$ and C has coordinates $(3,21)$, this line has slope $= \frac{21-9}{3-12} = \frac{12}{-9} = -\frac{4}{3}$.

Since the line has slope $-\frac{4}{3}$ and point $(3,21)$ lies on the line, we have
$\frac{y-21}{x-3} = -\frac{4}{3} \implies 3y - 63 = -4x + 12 \implies 3y = -4x + 75 \implies y = -\frac{4}{3}x + 25$.

Since $T(r,s)$ lies on this line, $s = -\frac{4}{3}r + 25$.

Using the distance formula, since $TD = 10$, we have

$$\begin{aligned} \sqrt{(r-12)^2+(s-9)^2} &= 10 \\ (r-12)^2+(s-9)^2 &= 100 \\ (r-12)^2 + \left(\left(-\frac{4}{3}r+25\right)-9\right)^2 &= 100,\ \text{since } s = -\frac{4}{3}r+25 \\ (r-12)^2 + \left(-\frac{4}{3}r+16\right)^2 &= 100 \\ r^2 - 24r + 144 + \frac{16}{9}r^2 - \frac{128}{3}r + 256 &= 100 \end{aligned}$$

$$\begin{aligned}
\frac{25}{9}r^2 - \frac{200}{3}r + 300 &= 0\\
\frac{25}{9}(r^2 - 24r + 108) &= 0\\
r^2 - 24r + 108 &= 0\\
(r-6)(r-18) &= 0\\
r &= 6,\ 18
\end{aligned}$$

But $r = 18$ lies outside the square. Therefore, $r = 6$ and $s = -\frac{4}{3}(6) + 25 = -8 + 25 = 17$.

The line l passes through $A(0,0)$ and $T(6,17)$, has y-intercept 0 and slope $= \frac{17-0}{6-0} = \frac{17}{6}$.

Therefore, the equation of line l is $y = \frac{17}{6}x$ or $17x - 6y = 0$.

Solution 2

Since $TD = 10$ and $CD = 15$, $CT = CD - TD = 15 - 10 = 5$.
$\triangle TDA$ is right-angled so, using the Pythagorean Theorem,

$$\begin{aligned}
AT^2 &= AD^2 + TD^2\\
(r-0)^2 + (s-0)^2 &= 15^2 + 10^2\\
r^2 + s^2 &= 325 \qquad (1)
\end{aligned}$$

Using the distance formula, we can calculate the length of each of CT and TD,

$$CT = \sqrt{(r-3)^2 + (s-21)^2} \qquad \text{and} \qquad TD = \sqrt{(r-12)^2 + (s-9)^2}$$

Squaring both sides and simplifying

$$CT^2 = r^2 - 6r + 9 + s^2 - 42s + 441 \qquad TD^2 = r^2 - 24r + 144 + s^2 - 18s + 81$$

Substituting $CT = 5$ and $TD = 10$

$$5^2 = r^2 + s^2 - 6r - 42s + 450 \qquad 10^2 = r^2 + s^2 - 24r - 18s + 225$$

Rearranging

$$6r + 42s = r^2 + s^2 + 425 \qquad 24r + 18s = r^2 + s^2 + 125$$

From (1), $r^2 + s^2 = 325$ so

$$6r + 42s = 325 + 425 \qquad 24r + 18s = 325 + 125$$

$$6r + 42s = 750 \quad (2) \qquad 24r + 18s = 450 \quad (3)$$

We now have a system of equations. Equation (3) subtract $4 \times$ equation (2) gives $-150s = -2550$ and $s = 17$ follows. Substituting for s in (2), we obtain $r = 6$.

The line l passes through $A(0,0)$ and $T(6,17)$, has y-intercept 0 and slope $= \frac{17-0}{6-0} = \frac{17}{6}$.

Therefore, the equation of line l is $y = \frac{17}{6}x$ or $17x - 6y = 0$.

For Further Thought:

The point U is on CB so that the area of $\triangle ABU$ = the area of $\triangle UAT$ = the area of $\triangle ATD$. Determine the coordinates of U. By finding U and T, you will have found two line segments, AU and AT, that divide square $ABCD$ into three equal areas.