Problem

The product of the integers from 1 to \(n\) can be written in abbreviated form as \(n!\) and we say “\(n\) factorial”. So \(n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1\).

For example, \(6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720\), and

\[11! = 11 \times 10 \times 9 \times \ldots \times 3 \times 2 \times 1 = 39,916,800.\]

Note that \(6!\) ends in 1 zero and \(11!\) ends in 2 zeros.

Determine the smallest possible value of \(n\) such that \(n!\) ends in exactly 1000 zeros.

Solution

When finding a solution to this problem, it may be helpful to work with possible values for \(n\) to determine the number of zeros that \(n!\) ends in. One could use a calculator as part of this but many calculators switch to scientific notation around 14!. A trial and error approach could work but it may be very time consuming. Our approach will be very systematic.

A zero is added to the end of a number when we multiply by 10. Multiplying a number by 10 is the same as multiplying a number by 2 and then by 5, or by 5 and then by 2, since \(2 \times 5 = 10\) and \(5 \times 2 = 10\).

So we want \(n\) to be the smallest integer such that the factorization of \(n!\) contains 1000 5s and 1000 2s. Every even number has a 2 in its factorization and every number that is a multiple of 5 has a 5 in its factorization. There are more numbers less than or equal to \(n\) that are multiples of 2 than multiples of 5. So once we find a number \(n\) such that \(n!\) has 1000 5s in its factorization, we can stop, we know that there will be a sufficient number of 2s in its factorization.

There are \(\left\lfloor \frac{n}{5} \right\rfloor\) numbers less than or equal to \(n\) that are divisible by 5. Note, the notation \(\lfloor x \rfloor\) means the floor of \(x\) and is the largest integer less than or equal to \(x\). So \(\lfloor 4.2 \rfloor = 4\), \(\lfloor 4.9 \rfloor = 4\) and \(\lfloor 4 \rfloor = 4\). Also, since \(5 \times 1000 = 5000\), we know that \(n \leq 5000\).

Numbers that are a multiple of 25 will add an additional factor of 5, since 25 = 5 × 5.

There are \(\left\lfloor \frac{n}{25} \right\rfloor\) numbers less than or equal to \(n\) that are divisible by 25.

Numbers that are a multiple of 125 will add an additional factor of 5, since 125 = 5 × 5 × 5 and two of the factors have already been counted when we looked at 5 and 25.

There are \(\left\lfloor \frac{n}{125} \right\rfloor\) numbers less than or equal to \(n\) that are divisible by 125.

Numbers that are a multiple of 625 will add an additional factor of 5, since 625 = 5 × 5 × 5 × 5 and three of the factors have already been counted when we looked at 5, 25 and 125.

There are \(\left\lfloor \frac{n}{625} \right\rfloor\) numbers less than or equal to \(n\) that are divisible by 625.

Numbers that are a multiple of 3125 will add an additional factor of 5, since 3125 = 5^5 and four of the factors have already been counted when we looked at 5, 25, 125 and 625.

There are \(\left\lfloor \frac{n}{3125} \right\rfloor\) numbers less than or equal to \(n\) that are divisible by 3125.
The next power of 5 to consider is $5^6 = 15,625$. But since $n \leq 5000$, we can stop.

So we know that $n$ must satisfy

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n}{125} \right\rfloor + \left\lfloor \frac{n}{625} \right\rfloor + \left\lfloor \frac{n}{3125} \right\rfloor = 1000$$

Let’s ignore the floor function. We know that $n$ is going to be close to satisfying

$$\frac{n}{5} + \frac{n}{25} + \frac{n}{125} + \frac{n}{625} + \frac{n}{3125} = 1000$$

$$\frac{625n}{3125} + \frac{125n}{3125} + \frac{25n}{3125} + \frac{5n}{3125} + \frac{n}{3125} = 1000$$

$$\frac{781n}{3125} = 1000$$

$$n = \frac{1000 \times 3125}{781}$$

$$n = 4001.2$$

How many zeros are at the end of $4001!$?

$$\left\lfloor \frac{4001}{5} \right\rfloor + \left\lfloor \frac{4001}{25} \right\rfloor + \left\lfloor \frac{4001}{125} \right\rfloor + \left\lfloor \frac{4001}{625} \right\rfloor + \left\lfloor \frac{4001}{3125} \right\rfloor = \left\lfloor 800.2 \right\rfloor + \left\lfloor 160.04 \right\rfloor + \left\lfloor 32.008 \right\rfloor + \left\lfloor 6.4016 \right\rfloor + \left\lfloor 1.28032 \right\rfloor$$

$$= 800 + 160 + 32 + 6 + 1$$

$$= 999$$

So the factorization of $4001!$ has 999 zeros at the end. We need 1 more factor of 5 in order to have 1000 zeros at the end. The first integer after 4001 to contain a factor of 5 is 4005.

Therefore, 4005 is the smallest number such that $4005!$ ends in 1000 zeros.

Indeed, we can check. The number of zeros at the end of $4004!$ is equal to the number of 5’s in its factorization, which is equal to

$$\left\lfloor \frac{4004}{5} \right\rfloor + \left\lfloor \frac{4004}{25} \right\rfloor + \left\lfloor \frac{4004}{125} \right\rfloor + \left\lfloor \frac{4004}{625} \right\rfloor + \left\lfloor \frac{4004}{3125} \right\rfloor = \left\lfloor 800.8 \right\rfloor + \left\lfloor 160.16 \right\rfloor + \left\lfloor 32.032 \right\rfloor + \left\lfloor 6.4064 \right\rfloor + \left\lfloor 1.28128 \right\rfloor$$

$$= 800 + 160 + 32 + 6 + 1$$

$$= 999$$

The number of zeros at the end of $4005!$ is equal to the number of 5’s in its factorization, which is equal to

$$\left\lfloor \frac{4005}{5} \right\rfloor + \left\lfloor \frac{4005}{25} \right\rfloor + \left\lfloor \frac{4005}{125} \right\rfloor + \left\lfloor \frac{4005}{625} \right\rfloor + \left\lfloor \frac{4005}{3125} \right\rfloor = \left\lfloor 801 \right\rfloor + \left\lfloor 160.2 \right\rfloor + \left\lfloor 32.04 \right\rfloor + \left\lfloor 6.408 \right\rfloor + \left\lfloor 1.2816 \right\rfloor$$

$$= 801 + 160 + 32 + 6 + 1$$

$$= 1000$$