Problem

The positive even integers 2 to 1600, inclusive, are each multiplied by the same positive integer, \( n \). All of the products are then added together and the resulting sum is a perfect square.

Determine the value of the smallest positive integer \( n \) that makes this true.

Solution

What does the prime factorization of a perfect square look like? Let’s look at a few examples: 
\( 9 = 3^2 \), \( 36 = 6^2 = 2^23^2 \), and \( 129600 = 360^2 = 2^65^23^4 \). Notice that the exponent on each of the prime factors in the prime factorization in each of the three examples is an even number.

The positive integer \( n \) is the smallest positive integer such that 
\[
2n + 4n + 6n + \cdots + 1596n + 1598n + 1600n
\]
(1)
is a perfect square.

Factoring (1), we obtain
\[
2n(1 + 2 + 3 + \cdots + 798 + 799 + 800)
= 2n \left( \frac{800 \times 801}{2} \right)
= n(800)(801)
= n(2^5)(5^2)(3^2)(89)
\]
(2)

In going from (2) to (3), we have expressed the \( 800 \times 801 \) as the product of prime factors.

We need to determine what additional factors are required to make the quantity in (3) a perfect square such that \( n \) is as small as possible. In order for the exponent on each prime in the prime factorization to be even, we need \( n \) to be \( 89 \times 2 = 178 \). Then the quantity in (3) becomes
\[
n(2^5)(5^2)(3^2)(89) = (2)(89)(2^5)(5^2)(3^2)(89) = (2^6)(5^2)(3^2)(89^2) = [(2^3)(5)(3)(89)]^2,
\]
a perfect square.

Therefore, the smallest positive integer is 178 and the perfect square is
\[
178 \times 800 \times 801 = 114062400 = (10680)^2.
\]