$25 $10

Problem of the Week
Problem D and Solution
Stack 'em High

Problem

You have as many red coins and blue coins as you need to complete this problem. Each red coin is worth $10 and each blue coin is worth $25. Using only red coins and blue coins, create stacks of coins so that each stack has a total value of $750. No two stacks can have the same number of red coins and each stack must contain at least one red coin and one blue coin. Determine the maximum number of different stacks of coins that can be made.

Solution

Solution 1

If no red coins were required, you would need $\$750 \div \$25 = 30$ blue coins. Therefore, at most $30 - 1 = 29$ blue coins would be required for any one stack.

If you were to use an odd number of blue coins, then the total value of the blue coins would be some number with ones digit 5. The value of the required red coins would then be $750 minus the value of the blue coins, producing a difference whose ones digit is also 5. But each red coin is worth $10 and no combination of red coins could produce a total whose ones digit is 5. Therefore, it is not possible to have an odd number of blue coins.

So the possible number of blue coins in any stack is an even integer between 1 and 29. There are 14 even numbers in this range. Since any even multiple of $25 produces a number whose units digit is 0 and $750 minus the value of the blue coins would then also has a units digit 0, there exists some number of red coins that will produce each possible difference.

Therefore, the maximum number of different stacks of coins is 14. (In the second solution the possibilities will be listed.)

Solution 2

In this solution, a more algebraic argument will be presented.

Let b represent the number of blue coins and r represent the number of red coins. Since each stack must contain at least 1 coin of each colour and we must use whole coins, both b and r are integers such that $b \geq 1$ and $r \geq 1$.

Since each blue coin is worth \$25, the total value of the blue coins is $25b$. Since each red coin is worth \$10, the total value of the red coins is $10r$. The total value of each stack is \$750 so $25b + 10r = 750$.

Dividing both sides of the equation by 5, we obtain $5b + 2r = 150$.   (1)

Rearranging equation (1) to isolate $5b$, we obtain $5b = 150 - 2r$.   (2)

Each term on the right side of equation (2) is even so the difference $150 - 2r$ will also be even. The value of the left side of equation (2) must then also be even. This can only be accomplished for even values of b. (An odd integer multiplied by 5 produces an odd integer but an even integer multiplied by 5 produces an even integer.)

Since each stack must contain at least one coin of each colour, letting $r = 1$ in equation (2) will generate the largest possible value of b. When $r = 1$, equation (2) becomes $5b = 150 - 2 = 148$ and $b = 29.6$. Since b is an integer, b is at most 29.

We now know that b is an even integer such that $1 \leq b \leq 29$. There are 14 even integers in this set of numbers, namely $2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26,$ and 28.

Although not required, we can calculate the corresponding r values for each of the b values.

Written as a set of ordered pairs (b, r), we have

$$(2,70),\ (4,65),\ (6,60),\ (8,55),\ (10,50),\ (12,45),\ (14,40)$$
$$(16,35), (18,30),\ (20,25),\ (22,20),\ (24,15),\ (26,10), \text{and } (28,5).$$

The restriction stating that no two stacks could have the same number of red coins has clearly been satisfied.