Problem of the Week
Problem D and Solution
Can You Repeat That A Little Later?

Problem

When $\dfrac{1}{70\ 000\ 000}$ is written as a decimal, what digit occurs in the 2018^{th} place after the decimal point?

Solution

Notice that $\dfrac{1}{70\ 000\ 000} = \dfrac{1}{10\ 000\ 000} \times \dfrac{1}{7} = 0.000\,000\,1 \times \dfrac{1}{7}$.

Also, note that $\frac{1}{7} = 0.\overline{142857}$. That is, when $\frac{1}{7}$ is written as a decimal, the digits after the decimal point occur in repeating blocks of the 6 digits 142857.

Therefore,

$$\frac{1}{70\ 000\ 000} = 0.000\,000\,1 \times \frac{1}{7} = 0.000\,000\,1 \times 0.\overline{142\,857} = 0.000\,000\,0\overline{1\,428\,57}.$$

That is, when $\frac{1}{70\ 000\ 000}$ is written as a decimal, the digits after the decimal point will be seven 0's followed by repeating blocks of the six digits 142 857.

We see the decimal representation of $\frac{1}{70\ 000\ 000}$ has the same repetition as that for $\frac{1}{7}$, but the pattern is shifted over 7 places. Therefore, the 2018^{th} digit after the decimal point when $\frac{1}{70\ 000\ 000}$ is written as a decimal is the same as the $(2018 - 7) = 2011^{\text{th}}$ digit after the decimal point when $\frac{1}{7}$ is written as a decimal.

Since $\dfrac{2011}{6} = 335\dfrac{1}{6}$, then the 2011^{th} digit after the decimal point occurs after 335 blocks of the repeating digits have been used. In 335 blocks of six digits, there are $335 \times 6 = 2010$ digits in total.

Therefore, the 2011^{th} digit is one digit into the 336^{th} block of repeating digits, so it must be a 1.

The 2018^{th} digit after the decimal point in the decimal representation of $\frac{1}{70\ 000\ 000}$ is the same as the 2011^{th} digit after the decimal point in the decimal representation of $\frac{1}{7}$ and is therefore a 1.