Problem of the Week
Problem D and Solution
Repetition By Product

Problem

A positive integer is to be placed in each box. Integers may be repeated, but the product of any four adjacent integers is always 120. Determine all possible values for \( x \).

Solution

In both solutions, let \( a_1 \) be the integer placed in the first box, \( a_2 \) the integer placed in the second box, \( a_4 \) the integer placed in the fourth box, and so on, as shown below.

\[
\begin{array}{cccccccc}
& a_1 & a_2 & 2 & a_4 & a_5 & 4 & x & a_{10} & a_{11} & 3 & a_{13} & a_{14} \\
\end{array}
\]

Solution 1

Consider boxes 3 to 6. Since the product of any four adjacent integers is 120, we have 
\( 2 \times a_4 \times a_5 \times 4 = 120 \). Therefore, \( a_4 \times a_5 = \frac{120}{2 \times 4} = 15 \). Since \( a_4 \) and \( a_5 \) are positive integers, there are 4 possibilities: \( a_4 = 1 \) and \( a_5 = 15 \), or \( a_4 = 15 \) and \( a_5 = 1 \), or \( a_4 = 3 \) and \( a_5 = 5 \), or \( a_4 = 5 \) and \( a_5 = 3 \).

In each of the four cases, we will have \( a_7 = 2 \). We can see why by considering boxes 4–7. We have \( a_4 \times a_5 \times 4 \times a_7 = 120 \), or \( 15 \times 4 \times a_7 = 120 \), since \( a_4 \times a_5 = 15 \). Therefore, \( a_7 = \frac{120}{15 \times 4} = 2 \).

Case 1; \( a_4 = 1 \) and \( a_5 = 15 \)
Consider boxes 5 to 8. We have \( a_5 \times 4 \times a_7 \times a_8 = 120 \), or \( 15 \times 4 \times 2 \times a_8 = 120 \), or \( a_8 = \frac{120}{15 \times 4 \times 2} = 1 \).
Next, consider boxes 6 to 9. We have \( 4 \times a_7 \times a_8 \times x = 120 \), or \( 4 \times 2 \times 1 \times x = 120 \), or \( x = \frac{120}{4 \times 2} = 15 \).
Let’s check that \( x = 15 \) satisfies the only other condition in the problem that we have not yet used, that is \( a_{12} = 3 \).
Consider boxes 9 to 12. If \( x = 15 \) and \( a_{12} = 3 \), then \( a_{10} \times a_{11} = \frac{120}{15 \times 3} = \frac{8}{3} \). But \( a_{10} \) and \( a_{11} \) must both be integers, so is not possible for \( a_{10} \times a_{11} = \frac{8}{3} \). Therefore, it must not be possible for \( a_4 = 1 \) and \( a_5 = 15 \), and so we find that there is no solution for \( x \) in this case.

Case 2; \( a_4 = 15 \) and \( a_5 = 1 \)
Consider boxes 5 to 8. We have \( a_5 \times 4 \times a_7 \times a_8 = 120 \), or \( 1 \times 4 \times 2 \times a_8 = 120 \), or \( a_8 = \frac{120}{4 \times 2} = 15 \).
Next, consider boxes 6 to 9. We have \( 4 \times a_7 \times a_8 \times x = 120 \), or \( x = \frac{120}{4 \times 2 \times 15} = 1 \).
Let’s check that \( x = 1 \) satisfies the only other condition in the problem that we have not yet used, that is \( a_{12} = 3 \).
Consider boxes 7 to 10. Since \( a_7 = 2 \), \( a_8 = 15 \) and \( x = 1 \), then \( a_{10} = \frac{120}{2 \times 15 \times 1} = 4 \). Similarly, \( a_{11} = \frac{120}{15 \times 4 \times 1} = 2 \). Then we have \( x \times a_{10} \times a_{11} \times a_{12} = 1 \times 4 \times 2 \times 3 = 24 \neq 120 \). Therefore, it must not be possible for \( a_4 = 15 \) and \( a_5 = 1 \). There is no solution for \( x \) in this case.
Case 3: $a_4 = 3$ and $a_5 = 5$
Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $5 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{5 \times 4 \times 2} = 3$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4 \times 3 \times 5} = 5$.
Let’s check that $x = 5$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 7 to 10. Since $a_7 = 2$, $a_8 = 3$ and $x = 5$, then $a_{10} = \frac{120}{2 \times 3 \times 5} = 4$. Similarly, $a_{11} = \frac{120}{3 \times 5 \times 4} = 2$. Then we have $x \times a_{10} \times a_{11} \times a_{12} = 5 \times 4 \times 2 \times 3 = 120$. Therefore, the condition that $a_{12} = 3$ is satisfied in the case where $a_4 = 3$ and $a_5 = 5$. If we continue to fill out the entries in the boxes, we obtain the entries shown in the diagram below.

\[
\begin{array}{cccccccccccccc}
5 & 4 & 2 & 3 & 5 & 4 & 2 & 3 & 5 & 4 & 2 & 3 & 5 & 4 \\
\end{array}
\]

We see that $x = 5$ is a possible solution. However, is it the only solution? We have one final case to check.

Case 4: $a_4 = 5$ and $a_5 = 3$
Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $3 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{3 \times 4 \times 2} = 5$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4 \times 3 \times 5} = 3$.
Let’s check that $x = 3$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 7 to 10. If $x = 3$ and $a_{12} = 3$, then $a_{10} \times a_{11} = \frac{120}{3 \times 3} = \frac{40}{3}$. But $a_{10}$ and $a_{11}$ must both be integers, so it is not possible for $a_{10} \times a_{11} = \frac{40}{3}$. Therefore, it must not be possible for $a_4 = 5$ and $a_5 = 3$, and so we find that there is no solution for $x$ in this case.

Therefore, the only possible value for $x$ is $x = 5$.

Solution 2
You may have noticed a pattern for the $a_i$'s in Solution 1. We will explore this pattern.

\[
\begin{array}{cccccccccccccc}
a_1 & a_2 & 2 & a_4 & a_5 & 4 & a_7 & a_8 & x & a_{10} & a_{11} & 3 & a_{13} & a_{14} \\
\end{array}
\]

Since the product of any four integers is 120, $a_1 a_2 a_3 a_4 = a_2 a_3 a_4 a_5 = 120$. Since both sides are divisible by $a_2 a_3 a_4$, and each is a positive integer, then $a_1 = a_5$.
Similarly, $a_2 a_3 a_4 a_5 = a_3 a_4 a_5 a_6 = 120$, and so $a_2 = a_6$.
In general, $a_n a_{n+1} a_{n+2} a_{n+3} = a_{n+1} a_{n+2} a_{n+3} a_{n+4}$, and so $a_n = a_{n+4}$.
We can use this along with the given information to fill out the boxes as follows:

\[
\begin{array}{cccccccccccccc}
x & 4 & 2 & 3 & x & 4 & 2 & 3 & x & 4 & 2 & 3 & x & 4 \\
\end{array}
\]

Therefore, $4 \times 2 \times 3 \times x = 120$ and so $x = \frac{120}{4 \times 2 \times 3} = 5$. 

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\text{QR Code}
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