Problem of the Week
Problem D and Solution
Maximize the Area

Problem
Two rectangles, $ABJH$ and $JDEF$, with integer side lengths, share a common corner at $J$ such that $HJD$ and $BJF$ are perpendicular line segments. The two rectangles are enclosed by a larger rectangle $ACEG$, as shown. The area of rectangle $ABJH$ is $6$ cm$^2$ and the area of rectangle $JDEF$ is $15$ cm$^2$. Determine the largest possible area of the rectangle $ACEG$. Note that the diagram is not intended to be to scale.

Solution
Let $AB = x$, $AH = y$, $JD = a$ and $JF = b$.

Therefore,

\[
\begin{align*}
AB &= HJ = GF = x, \\
AH &= BJ = CD = y, \\
BC &= JD = FE = a, \text{ and} \\
HG &= JF = DE = b.
\end{align*}
\]

Then area$(ACEG) = \text{area}(ABJH) + \text{area}(BCDJ) + \text{area}(JDEF) + \text{area}(HJFG)$

\[
= 6 + ya + 15 + xb
\]

\[
= 21 + ya + xb
\]

Since the area of rectangle $ABJH$ is $6$ cm$^2$ and the side lengths of $ABJH$ are integers, then the side lengths must be $1$ and $6$ or $2$ and $3$. That is, $x = 1$ cm and $y = 6$ cm, $x = 6$ cm and $y = 1$ cm, $x = 2$ cm and $y = 3$ cm, or $x = 3$ cm and $y = 2$ cm.

Since the area of rectangle $JDEF$ is $15$ cm$^2$ and the side lengths of $JDEF$ are integers, then the side lengths must be $1$ and $15$ or $3$ and $5$. That is, $a = 1$ cm and $b = 15$ cm, $a = 15$ cm and $b = 1$ cm, $a = 3$ cm and $b = 5$ cm, or $a = 5$ cm and $b = 3$ cm.

To maximize the area, we need to pick the values of $x, y, a, b$ which make $ya + xb$ as large as possible. We will now break into cases based on the possible side lengths of $ABJH$ and $JDEF$ and calculate the area of $ACEG$ in each case. We do not need to try all $16$ possible pairings, because trying $x = 1$ cm and $y = 6$ cm with the four possibilities of $a$ and $b$ will give the same $4$ areas, in some order, as trying $x = 6$ cm and $y = 1$ cm with the four possibilities of $a$ and $b$. Similarly, trying $x = 2$ cm and $y = 3$ cm with the four possibilities of $a$ and $b$ will give the same $4$ areas, in some order, as trying $x = 3$ cm and $y = 2$ cm with the four possibilities of $a$ and $b$. (As an extension, we will leave it to you to think about why this is the case.)
Case 1: $x = 1$ cm, $y = 6$ cm and $a = 1$ cm, $b = 15$ cm
\[
\text{area}(ACEG) = 21 + ya + xb = 21 + 6(1) + 1(15) = 42 \text{ cm}^2
\]

Case 2: $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm
\[
\text{area}(ACEG) = 21 + ya + xb = 21 + 6(15) + 1(1) = 112 \text{ cm}^2
\]

Case 3: $x = 1$ cm, $y = 6$ cm and $a = 3$ cm, $b = 5$ cm
\[
\text{area}(ACEG) = 21 + ya + xb = 21 + 6(3) + 1(5) = 44 \text{ cm}^2
\]

Case 4: $x = 1$ cm, $y = 6$ cm and $a = 5$ cm, $b = 3$ cm
\[
\text{area}(ACEG) = 21 + ya + xb = 21 + 6(5) + 1(3) = 54 \text{ cm}^2
\]

Case 5: $x = 2$ cm, $y = 3$ cm and $a = 1$, $b = 15$ cm
\[
\text{area}(ACEG) = 21 + ya + xb = 21 + 3(1) + 2(15) = 54 \text{ cm}^2
\]

Case 6: $x = 2$ cm, $y = 3$ cm and $a = 15$, $b = 1$ cm
\[
\text{area}(ACEG) = 21 + ya + xb = 21 + 3(15) + 2(1) = 68 \text{ cm}^2
\]

Case 7: $x = 2$ cm, $y = 3$ cm and $a = 3$, $b = 5$ cm
\[
\text{area}(ACEG) = 21 + ya + xb = 21 + 3(3) + 2(5) = 40 \text{ cm}^2
\]

Case 8: $x = 2$ cm, $y = 3$ cm and $a = 5$, $b = 3$ cm
\[
\text{area}(ACEG) = 21 + ya + xb = 21 + 3(5) + 2(3) = 42 \text{ cm}^2
\]

We see that the maximum area is $112$ cm$^2$, and occurs when $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm. It will also occur when $x = 6$ cm, $y = 1$ cm and $a = 1$ cm, $b = 15$ cm.

The following diagrams show the calculated values placed on the original diagram. The diagram was definitely not drawn to scale! Both solutions produce rectangles with dimensions $7$ cm by $16$ cm, and area $112 \text{ cm}^2$. 

![Diagram 1](image1.png)

![Diagram 2](image2.png)

![Diagram 3](image3.png)