



Problem of the Week

Problem C and Solution

Gooooooooallllll!

Problem

The Beaver Rangers played a soccer game against the Fox Raiders. A total of 6 goals were scored in the game. Neither team scored three goals in a row at any point during the game. Determine all possible final scores and all of the different ways each score could have been obtained so that the Beaver Rangers won the game.

Solution

In order to win, the Beavers Rangers would have to score more goals than the Fox Raiders. We will use B to represent a goal scored by the Beavers and an F to represent a goal scored by the Foxes. Since a total of 6 goals were scored, the only possibilities for winning scores for the Beavers could be $4 - 2$, $5 - 1$ and $6 - 0$. Not all scores were possible since we know that neither team scored more than two goals in a row.

- Was a $6 - 0$ Beaver victory possible?

We can easily eliminate $6 - 0$ since the Beavers would have had to score more than 2 consecutive goals.

- Was a $5 - 1$ Beaver victory possible?

In a $5 - 1$ victory for the Beaver Rangers, the Fox Raiders would have only scored 1 goal. Consider placing 5 Bs and 1 F in 6 spaces so that there are no more than 2 consecutive Bs? No matter where you place the F, there will always be at least 3 consecutive Bs. This is illustrated below. \boxtimes indicates that a possibility is not valid.

1 st goal scored by Foxes:	<u>F</u> _ _ _ _ _	5 Bs in a row. \boxtimes
2 nd goal scored by Foxes:	_ <u>F</u> _ _ _ _	4 Bs in a row. \boxtimes
3 rd goal scored by Foxes:	_ _ <u>F</u> _ _ _	3 Bs in a row. \boxtimes
4 th goal scored by Foxes:	_ _ _ <u>F</u> _ _	3 Bs in a row. \boxtimes
5 th goal scored by Foxes:	_ _ _ _ <u>F</u> _	4 Bs in a row. \boxtimes
6 th goal scored by Foxes:	_ _ _ _ _ <u>F</u>	5 Bs in a row. \boxtimes

Under the conditions given in the problem, it is not possible for the Beaver Rangers to win with a $5 - 1$ score.

- Was a $4 - 2$ Beaver victory possible?

In a $4 - 2$ victory for the Beaver Rangers, the Fox Raiders would have scored 2 goals. These goals could have been consecutive or they could have been separated by at least one goal by the Beavers. We will examine these two possibilities separately.





i) Was a 4 – 2 Beaver victory possible if the Foxes scored two consecutive goals?

This would only have been possible if the Foxes had scored the third and fourth goals. Consider placing 4 Bs and 2 Fs in 6 spaces so that there are no more than 2 consecutive Bs and the 2 Fs are together? The possibilities are illustrated below. \boxtimes indicates that a possibility is not valid. \checkmark indicates that a possibility is valid.

- 1st and 2nd goals scored by Foxes: F F 4 Bs in a row. \boxtimes
- 2nd and 3rd goals scored by Foxes: F F 3 Bs in a row. \boxtimes
- 3rd and 4th goals scored by Foxes: F F 2 Bs in a row. \checkmark
- 4th and 5th goals scored by Foxes: F F 3 Bs in a row. \boxtimes
- 5th and 6th goals scored by Foxes: F F 4 Bs in a row. \boxtimes

Under the conditions given in the problem, the only way the Beaver Rangers won 4 – 2 when the Foxes scored two consecutive goals occurred when the Foxes scored the third and fourth goals and the Beavers scored the remaining four goals.

ii) Was a 4 – 2 Beaver victory possible if the Foxes never scored two consecutive goals?

The possibilities are illustrated below. \boxtimes indicates that a possibility is not valid. \checkmark indicates that a possibility is valid.

- 1st and 3rd goals scored by Foxes: F F 3 Bs in a row. \boxtimes
- 1st and 4th goals scored by Foxes: F F 2 Bs in a row. \checkmark
- 1st and 5th goals scored by Foxes: F F 3 Bs in a row. \boxtimes
- 1st and 6th goals scored by Foxes: F F 4 Bs in a row. \boxtimes
- 2nd and 4th goals scored by Foxes: F F 2 Bs in a row. \checkmark
- 2nd and 5th goals scored by Foxes: F F 2 Bs in a row. \checkmark
- 2nd and 6th goals scored by Foxes: F F 3 Bs in a row. \boxtimes
- 3rd and 5th goals scored by Foxes: F F 2 Bs in a row. \checkmark
- 3rd and 6th goals scored by Foxes: F F 2 Bs in a row. \checkmark
- 4th and 6th goals scored by Foxes: F F 3 Bs in a row. \boxtimes

Under the conditions given in the problem, there were five ways for the Beaver Rangers to win with a 4 – 2 score when the Foxes never scored two consecutive goals. These possibilities occurred when the Foxes scored the first and fourth goals or the second and fourth goals or the second and fifth goals or the third and fifth goals or the third and sixth goals. In each of these instances, the Beavers scored all of the remaining goals.

Under the conditions of the problem, a score of 4 – 2 for the Beavers was the only possible winning score. This score could have been obtained six different ways. The scoring possibilities are summarized in the following table.

1 st Goal	2 nd Goal	3 rd Goal	4 th Goal	5 th Goal	6 th Goal
B	B	F	F	B	B
F	B	B	F	B	B
B	F	B	F	B	B
B	F	B	B	F	B
B	B	F	B	F	B
B	B	F	B	B	F

