Problem of the Week
Problem C and Solution
Rotate Right, Rotate Left

Problem
Two equilateral triangles, \( \triangle ABC \) and \( \triangle PQR \), have their bases \( AB \) and \( PQ \) sitting on line segment \( MN \), as shown.

\( \triangle ABC \) is tipped clockwise 65° about point \( B \) so that \( \angle MBA = 65° \) and point \( B \) remains where it is on \( MN \). \( \triangle PQR \) is tipped counterclockwise 75° about point \( P \) so that \( \angle NPQ = 75° \) and point \( P \) remains where it is on \( MN \). As a result of tipping the two, \( \triangle ABC \) overlaps \( \triangle PQR \) such that \( AC \) and \( BC \) intersect \( RP \) at \( X \) and \( Y \), respectively. Vertex \( C \) of \( \triangle ABC \) lies inside \( \triangle PQR \). Determine the measure of \( \angle CXY \).

Solution
In any equilateral triangle, all sides are equal in length and each angle measures 60°.

Since \( \triangle ABC \) and \( \triangle PQR \) are equilateral,
\[ \angle ABC = \angle ACB = \angle CBA = \angle QPR = \angle PRQ = \angle RQP = 60°. \]

Since the angles in a straight line sum to 180°, we have
\[ 180° = 65° + \angle ABC + \angle YBP = 65° + 60° + \angle YBP. \]
Rearranging, we have \( \angle YBP = 180° - 65° - 60° = 55°. \)

Similarly, since angles in a straight line sum to 180°, we have
\[ 180° = 75° + \angle QPR + \angle YPB = 75° + 60° + \angle YPB. \]
Rearranging, we have \( \angle YPB = 180° - 75° - 60° = 45°. \)

Since the angles in a triangle sum to 180°, in \( \triangle BYP \) we have
\[ \angle YPB + \angle YBP + \angle BYP = 180°, \]
and so \( 45° + 55° + \angle BYP = 180°. \)
Rearranging, we have \( \angle BYP = 180° - 45° - 55° = 80°. \)

When two lines intersect, vertically opposite angles are equal. Since \( \angle XYC \) and \( \angle BYP \) are vertically opposite angles, we have \( \angle XYC = \angle BYP = 80°. \)

Again, since angles in a triangle sum to 180°, in \( \triangle XYC \) we have
\[ \angle XYC + \angle XCY + \angle CXY = 180°. \]
We have already found that \( \angle XYC = 80° \), and since \( \angle XCY = \angle ACB \), we have \( \angle XCY = 60°. \) So, \( \angle XYC + \angle XCY + \angle CXY = 180° \) becomes \( 80° + 60° + \angle CXY = 180°. \) Rearranging, we have \( \angle CXY = 180° - 80° - 60° = 40°. \)

Therefore, \( \angle CXY = 40°. \)