



Problem of the Week

Problem E and Solution

Clipping Along



Problem

Gwen and Chris are playing a game. They begin with a pile of paperclips, and use the following rules.

1. The two players alternate turns.
2. On any turn, a player can remove 1, 2, 3, 4, or 5 paperclips from the pile.
3. The same number of paperclips cannot be removed on two different turns during the entire game.
4. The last person who is able to play wins, regardless of whether there are any paperclips remaining in the pile after their turn.

Suppose the game starts with 8 paperclips and Gwen goes first. Find all initial moves that Gwen can make to guarantee that she will win. Justify your answers.

Solution

Gwen has five options to begin with. She can remove 5, 4, 3, 2, or 1 paperclip. Let's look at each case.

Case 1– Gwen removes 5 paperclips on her first turn.

If Gwen removes 5 paperclips then Chris can remove the remaining 3 paperclips on his turn. Chris would win the game. Therefore, Gwen could lose if she starts by removing 5 paperclips.

Case 2 – Gwen removes 4 paperclips on her first turn.

If Gwen removes 4 paperclips, then there are 4 paperclips remaining. Chris cannot remove 4 paperclips or he will violate the third rule. So, Chris can only remove 1, 2 or 3 paperclips.

If Chris removes 1 paperclip, then there are 3 paperclips remaining. Gwen can now remove the remaining 3 paperclips on her second turn to win the game.

If Chris removes 2 paperclips, then there are 2 paperclips remaining. Gwen can now remove 1 paperclip on her second turn. This leaves 1 paperclip which Chris cannot remove without violating the third rule. Gwen wins the game.

If Chris removes 3 paperclips, then there is 1 paperclip remaining. Gwen can now remove the remaining paper clip on her second turn to win the game.

Therefore, no matter what Chris does on his first turn, Gwen can win if she starts the game by removing 4 paperclips.

Case 3 – Gwen removes 3 paperclips on her first turn.

If Gwen removes 3 paperclips then Chris can remove the remaining 5 paperclips to win the game. Therefore, Gwen could lose if she starts by removing 3 paperclips.





Case 4 – Gwen removes 2 paperclips on her first turn.

If Gwen removes 2 paperclips, then there are 6 paperclips remaining. Chris cannot remove 2 paperclips without violating the third rule. So, Chris can remove 1, 3, 4, or 5 paperclips.

If Chris removes 1 paperclip, then Gwen can remove the 5 remaining paperclips on her second turn to win the game.

If Chris removes 3 paperclips then 3 paperclips remain. Gwen can remove 1 paperclip on her second turn. (Gwen cannot remove 2 or 3 paperclips without violating the third rule.) There are 2 paperclips left. Chris would lose since he cannot remove 1 or 2 paperclips without violating the third rule. Therefore, Gwen wins the game.

If Chris removes 4 paperclips, then 2 paperclips remain. Gwen will remove 1 paperclip on her second turn leaving 1 paperclip. Chris would lose since he cannot remove 1 paperclip without violating the third rule. Therefore, Gwen wins the game.

If Chris removes 5 paperclips, then Gwen can remove the remaining paperclip on her second turn to win the game.

Therefore, no matter what Chris does on his first turn, Gwen can win if she starts the game by removing 2 paperclips.

Case 5 – Gwen removes 1 paperclip on her first turn.

If Gwen removes 1 paperclip, then there are 7 paperclips remaining. Chris can then remove 2, 3, 4, or 5 paperclips.

If Chris removes 2 paperclips, then Gwen can remove the remaining 5 paperclips on her second turn to win the game.

If Chris removes 3 paperclips, then Gwen can remove the remaining 4 paperclips on her second turn to win the game.

If Chris removes 4 paperclips, then Gwen can remove the remaining 3 paperclips on her second turn to win the game.

If Chris removes 5 paperclips, then Gwen can remove the remaining 2 paperclips on her second turn to win the game.

Therefore, no matter what Chris does on his first turn, Gwen can win if she starts the game by removing 1 paperclip.

Therefore, Gwen can win the game if she goes first and starts by removing 1, 2 or 4 paperclips on her first turn.

