Problem of the Week

Problem E and Solution

Useful Facts Indeed!

Problem $$2^{21609d} - 1 = 746\ 093\ 103\ 064\ 661\ 343\ \cdots\ 7$$

Some number 21609d, with units digit d, $2^{21609d} - 1$ is a very large prime number. In fact, the number contains 65 050 digits. The number begins 746 093 103 064 661 343 $\cdots$ and ends with the units digit 7. Determine the value of d, the units digit of 21609d.

Here are some useful facts which may be helpful in solving this problem:

- if n is divisible by 3, then $2^n - 1$ is divisible by 7; and
- if n is divisible by 5, then $2^n - 1$ is divisible by 31.

Solution

To start, let's look for a pattern in the units digit of powers of 2.

$2^1 = 2$	$2^2 = 4$	$2^3 = 8$	$2^4 = 16$
$2^5 = 32$	$2^6 = 64$	$2^7 = 128$	$2^8 = 256$

It appears that the units digit of powers of 2 repeat in the cycle 2, 4, 8, 6. The next four powers of 2, 2^9, 2^{10}, 2^{11}, and 2^{12}, end with units digits 2, 4, 8, and 6, respectively, as expected.

Then 2^{216088} would end in a 6 since 216088 is divisible by 4. It then follows that 2^{216089} ends in 2, 2^{216090} ends in 4 and 2^{216091} ends in an 8. Since 2^{216091} ends in an 8, 2^{216095} and 2^{216099} also ends in 8.

Then $2^{216091} - 1$, $2^{216095} - 1$ and $2^{216099} - 1$ each end in a 7. Therefore, the only possible values of d are 1, 5 and 9.

If a number ends in 0 or 5, then it is divisible by 5. If $d = 5$ then 216095 is divisible by 5. From the useful facts, it follows that $2^{216095} - 1$ is divisible by 31 and is therefore not a prime number.

If the sum of the digits of a number is divisible by 3, then the number is divisible by three. If $d = 9$ then 216099 is divisible by 3, since the sum of the digits of 216099 is 27 which is divisible by 3. From the useful facts, it follows that $2^{216099} - 1$ is divisible by 7 and is therefore not a prime number.

The only possible value for d is 1 and $2^{216091} - 1$ is a prime number ending in 7. This prime number is from a group of prime numbers called *Mersenne primes*. This number is the 31^{st} Mersenne prime and it was discovered in September of 1985. For more on Mersenne Primes, check out the Great Internet Mersenne Prime Search (GIMPS) at www.mersenne.org. According to GIMPS, as of January 2016, 49 Mersenne Primes are known. Perhaps you will be part of a team that will discover the next Mersenne Prime. There are prizes awarded when new discoveries are found and verified.

Extension: Can you prove the two useful facts?