Problem of the Week
Problem E and Solution
Doughnuts!

Problem

"Baker's Dozen Doughnut Shop" doughnuts are sold only in boxes of 7, 13, or 25. To buy 14 doughnuts you must order two boxes of 7, but you cannot buy exactly 15 doughnuts since no combination of boxes contains 15 doughnuts. What is the maximum number of doughnuts that *cannot* be ordered using combinations of the three different size boxes from "Baker's Dozen Doughnut Shop"?

Solution

We can fill any order size which is a multiple of 7. Therefore, we can fill orders for $\{7, 14, 21, 28, 35, 42, 49, \cdots\}$ doughnuts. We can also fill any order size which is a multiple of 13. Therefore, we can fill orders for $\{13, 26, 39, 52, \cdots\}$ doughnuts. And we can fill any order size which is a multiple of 25. Therefore, we can fill orders for $\{25, 50, 75, \cdots\}$ doughnuts.

Using the multiples above and combinations of the three different size boxes, we can fill orders of the following sizes:

$$7,\ 13,\ 14,\ 20\ (7+13),\ 21,\ 25,\ 26,\ 27\ (14+13),\ 28,\ 32\ (7+25),\ 33\ (7+26),$$
$$34\ (21+13),\ 35,\ 38\ (13+25),\ 39,\ 40\ (14+26),\ 41\ (28+13),\ 42,$$
$$45\ (7+13+25),\ 46\ (21+25),\ 47\ (21+26),\ 48\ (35+13),\ 49,\ 50,\ \text{and}\ 51\ (26+25).$$

The missing numbers from the above list correspond to the order sizes that cannot be filled. The largest order that we are unable to fill in the above list appears to be 44. But we must justify that this is the maximum order size which cannot be filled. To do this, we note that orders of sizes 45, 46, 47, 48, 49, 50 and 51 can all be filled. This corresponds to 7 consecutive order sizes. If we add a 7 - pak to each of these order sizes, we can fill the next seven consecutive order sizes. That is, we can fill orders of 52, 53, 54, 55, 56, 57 and 58. If we add a 7 - pak to each of these orders, we can fill the next seven consecutive order sizes. In fact, every order size of 45 or more doughnuts can be filled. Since an order of size 44 doughnuts cannot be filled, this is the maximum size order which cannot be filled.

It turns out that there are only 26 order sizes that "Baker's Dozen Doughnut Shop" cannot fill using the three different size boxes of doughnuts. If the shop were to add a box containing 3 doughnuts, how many orders would be impossible to fill?