Problem of the Week
Problem E and Solution
Paper Folding 101

Problem
The problem today involves simple paper folding. In fact, only one fold is required.

A rectangular piece of paper is 30 cm wide and 40 cm long. The paper has a pattern on one
side and is plain on the other. The paper is folded so that the two diagonally opposite corners,
A and C, coincide. (This is illustrated on the diagram to the right.)

Determine the length of the crease, $FE$, created by the fold.

Solution
After the fold, $C$ coincides with $A$ and $D$ folds to $G$. The angle at $G$ is the same as the angle
at $D$. Since $ABCD$ is a rectangle, $\angle ADC = 90^\circ$ and it follows that $\angle AGF = 90^\circ$.

Let $a$ represent the length of $BE$ and $b$ represent the length of $FD$.
Then $EC = CB - BE = 40 - a$ and $AF = AD - FD = 40 - b$.

The distance from the top of the crease at $F$ to $D$ is the same length as the distance from $F$ to
$G$. It follows that $FG = FD = b$.

The distance from the bottom of the crease at $E$ to $C$ is the same length as the distance from
$E$ to $A$. It follows that $AE = EC = 40 - a$.

All of the information is recorded on the following diagram.
Since $\triangle ABE$ and $\triangle AGF$ are both right angled, we can use the Pythagorean Theorem to find $a$ and $b$.

\[
\begin{align*}
BE^2 + AB^2 &= AE^2 \\
&= a^2 + 30^2 = (40 - a)^2 \\
a^2 + 900 &= 1600 - 80a + a^2 \\
80a &= 700 \\
a &= \frac{35}{4}
\end{align*}
\]

\[
\begin{align*}
FG^2 + AG^2 &= AF^2 \\
&= b^2 + 30^2 = (40 - b)^2 \\
b^2 + 900 &= 1600 - 80b + b^2 \\
80b &= 700 \\
b &= \frac{35}{4}
\end{align*}
\]

\[
\therefore a = b = \frac{35}{4}
\]

We still need to find the length of the crease.

From $F$ drop a perpendicular to $BC$ intersecting at $H$. $FHCD$ is a rectangle. It follows that $FH = DC = 30$ and $HC = FD = b$.

Also, $EH = BC - BE - HC = 40 - a - b = 40 - \frac{35}{4} - \frac{35}{4} = \frac{90}{4} = \frac{45}{2}$.

Using the Pythagorean Theorem in $\triangle EFH$,

\[
\begin{align*}
EF^2 &= FH^2 + EH^2 \\
&= 30^2 + \left(\frac{45}{2}\right)^2 \\
&= 900 + \frac{2025}{4} \\
&= \frac{3600 + 2025}{4} \\
&= \frac{5625}{4} \\
EF &= \frac{75}{2} \quad (EF > 0)
\end{align*}
\]

The length of the crease is $\frac{75}{2}$ cm (37.5 cm).