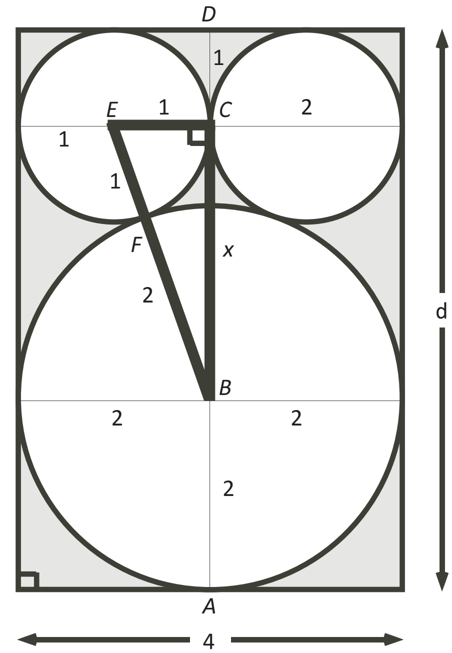
Problem of the Week
Problem E and Solution
This Picture Looks Like ...



Problem

The shape of the head and ears of a famous mouse appears to be contained in a rectangle. The two smaller circles have equal radii. Each of the three circles is tangent to the other two circles, and each is also tangent to the sides of the rectangle. The width of the rectangle is 4 m.

Solution

Since the larger circle is tangent to two opposite sides of the rectangle, its diameter is 4 m, the width of the rectangle. It follows that the radius of the larger circle is 2 m.

The two smaller circles have equal radii, are tangent to each other and to opposite sides of the rectangle. It follows that the diameter of each of the smaller circles is half the width of the rectangle, namely 2 m. The radius of each of the smaller circles is 1 m.

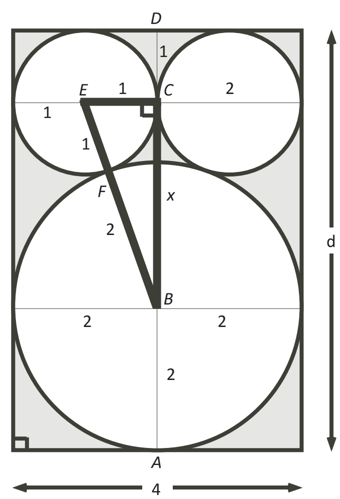
Let the centre of the large circle and leftmost small circle be B and E respectively. Let the two small circles be tangent at C. Let the leftmost small circle and the larger circle be tangent at F. Position line segment AD so that it is parallel to the longer side such that A and D are midpoints of the shorter sides of the rectangle. AD will pass through C and B.

Let the length of the rectangle be d. This is the same as the distance from A to D on the diagram. We know that $AB = 2$ m, the radius of the larger circle, and $CD = 1$ m, the radius of the smaller circle. We need to find the length of BC.

AD is tangent to the smaller circles at C. Using the first property, we know that $EC \perp AD$ at C. Using the second property, EFB is a straight line segment and $EB = EF + FB = 1 + 2 = 3$ m.

Combining the information, $\triangle ECB$ is right angled at C. Using the Pythagorean Theorem, $BC^2 = EB^2 - EC^2 = 3^2 - 1^2 = 8$ and $BC = \sqrt{8}$ follows. Then the length of the rectangle is

$$d = AB + BC + CD = 2 + \sqrt{8} + 1 = 3 + \sqrt{8}.$$



To find the area not covered by the head and ears we need to find the shaded area. To do this we find the area of the rectangle and subtract the area of the large circle and the area of the two equal radii smaller circles.

$$\begin{aligned} & \text{Shaded Area} \\ = \ & \text{Area of Rectangle} - \text{Area of Large Circle} - \text{Area of two smaller circles} \\ = \ & 4 \times (3 + \sqrt{8}) - \pi \times 2^2 - 2 \times (\pi \times 1^2) \\ = \ & 12 + 4\sqrt{8} - 4\pi - 2\pi \\ = \ & 12 + 4\sqrt{8} - 6\pi \end{aligned}$$

Some students have learned to simplify radicals and know that $\sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$. The shaded area can then be written $12 + 4 \times 2\sqrt{2} - 6\pi = 12 + 8\sqrt{2} - 6\pi$.

The shaded area is $(12 + 4\sqrt{8} - 6\pi)$ m^2 or $(12 + 8\sqrt{2} - 6\pi)$ m^2 (approximately 4.5 m^2).